6 Exponential Functions and Sequences

- 6.1 Exponential Functions
- 6.2 Exponential Growth and Decay
- 6.3 Comparing Linear and Exponential Functions
- 6.4 Solving Exponential Equations
- 6.5 Geometric Sequences
- 6.6 Recursively Defined Sequences

SEE the Big Idea

- Fibonacci and Flowers (p. 317)
- Bacterial Culture (p. 304)
- Coyote Population (p. 279)
- Soup Kitchen (p. 312)
- Town Population (p. 293)
Maintaining Mathematical Proficiency

Using Order of Operations

Example 1 Evaluate $10^2 \div (30 \div 3) - 4(3 - 9) + 5^1$.

First: Parentheses $10^2 \div (30 \div 3) - 4(3 - 9) + 5^1 = 10^2 \div 10 - 4(-6) + 5$

Second: Exponents $= 10 \div 10 - 4(-6) + 5$

Third: Multiplication and Division (from left to right) $= 10 + 24 + 5$

Fourth: Addition and Subtraction (from left to right) $= 39$

Evaluate the expression.

1. $12\left(\frac{14}{2}\right) - 3^3 + 15 - 9^2$
2. $5^2 \cdot 8 + 2^2 + 20 \cdot 3 - 4$
3. $-7 + 16 + 2^4 + (10 - 4^2)$

Zero and Negative Exponents

Example 2 Evaluate $4^{-3}$.

$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

Example 3 Evaluate $4 - 5 \cdot 3^0$.

$4 - 5 \cdot 3^0 = 4 - 5 \cdot 1 = 4 - 5 = -1$

Evaluate the expression.

4. $2^{-5}$
5. $-5^{-3}$
6. $(-2)^{-1} \cdot 6^0$
7. $7^0 + 8^2 \div 4$

Writing Equations for Arithmetic Sequences

Example 4 Write an equation for the $n$th term of the arithmetic sequence 5, 15, 25, 35, . . .

The first term is 5, and the common difference is 10.

$$a_n = a_i + (n - 1)d$$

Equation for an arithmetic sequence

$$a_n = 5 + (n - 1)(10)$$

Substitute 5 for $a_i$ and 10 for $d$.

$$a_n = 10n - 5$$

Simplify.

Write an equation for the $n$th term of the arithmetic sequence.

8. 12, 14, 16, 18, . . .
9. 6, 3, 0, −3, . . .
10. 22, 15, 8, 1, . . .

11. ABSTRACT REASONING Let $a$ be a nonzero number and let $n$ be a positive integer. Explain why $\frac{1}{a^{-n}} = a^n$. 
Mathematical Practices

Problem-Solving Strategies

Core Concept

Finding a Pattern
When solving a real-life problem, look for a pattern in the data. The pattern could include repeating items, numbers, or events. After you find the pattern, describe it and use it to solve the problem.

Example 1 Using a Problem-Solving Strategy

The volumes of seven chambers of a chambered nautilus are given. Find the volume of Chamber 10.

**SOLUTION**

To find a pattern, try dividing each volume by the volume of the previous chamber.

<table>
<thead>
<tr>
<th>Chamber</th>
<th>Volume (cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.836</td>
</tr>
<tr>
<td>2</td>
<td>0.889</td>
</tr>
<tr>
<td>3</td>
<td>0.945</td>
</tr>
<tr>
<td>4</td>
<td>1.005</td>
</tr>
<tr>
<td>5</td>
<td>1.068</td>
</tr>
<tr>
<td>6</td>
<td>1.135</td>
</tr>
<tr>
<td>7</td>
<td>1.207</td>
</tr>
</tbody>
</table>

From this, you can see that the volume of each chamber is about 6.3% greater than the volume of the previous chamber. To find the volume of Chamber 10, multiply the volume of Chamber 7 by 1.063 three times.

\[
1.207 \times 1.063 \approx 1.283
\]
\[
1.283 \times 1.063 \approx 1.364
\]
\[
1.364 \times 1.063 \approx 1.450
\]

The volume of Chamber 10 is about 1.450 cubic centimeters.

Monitoring Progress

1. A rabbit population over 8 consecutive years is given by 50, 80, 128, 205, 328, 524, 839, 1342. Find the population in the tenth year.

2. The sums of the numbers in the first eight rows of Pascal’s Triangle are 1, 2, 4, 8, 16, 32, 64, 128. Find the sum of the numbers in the tenth row.
6.1 Exponential Functions

Essential Question What are some of the characteristics of the graph of an exponential function?

Exploration 1 Exploring an Exponential Function

Work with a partner. Copy and complete each table for the exponential function \( y = 16(2)^x \). In each table, what do you notice about the values of \( x \)? What do you notice about the values of \( y \)?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 16(2)^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>2</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>128</td>
</tr>
<tr>
<td>4</td>
<td>256</td>
</tr>
<tr>
<td>5</td>
<td>512</td>
</tr>
</tbody>
</table>

Exploration 2 Exploring an Exponential Function

Work with a partner. Repeat Exploration 1 for the exponential function \( y = 16(\frac{1}{2})^x \). Do you think the statement below is true for any exponential function? Justify your answer.

“As the independent variable \( x \) changes by a constant amount, the dependent variable \( y \) is multiplied by a constant factor.”

Exploration 3 Graphing Exponential Functions

Work with a partner. Sketch the graphs of the functions given in Explorations 1 and 2. How are the graphs similar? How are they different?

Communicate Your Answer

4. What are some of the characteristics of the graph of an exponential function?

5. Sketch the graph of each exponential function. Does each graph have the characteristics you described in Question 4? Explain your reasoning.

a. \( y = 2^x \)  
   b. \( y = 2(3)^x \)  
   c. \( y = 3(1.5)^x \)  
   d. \( y = \left(\frac{1}{2}\right)^x \)  
   e. \( y = 3\left(\frac{1}{2}\right)^x \)  
   f. \( y = 2\left(\frac{3}{4}\right)^x \)
6.1 Lesson

What You Will Learn

- Identify and evaluate exponential functions.
- Graph exponential functions.
- Solve real-life problems involving exponential functions.

Core Vocabulary

- exponential function, p. 274

Previous

- independent variable
- dependent variable
- parent function

The exponential function is a nonlinear function of the form \( y = ab^x \), where \( a \neq 0 \), \( b \neq 1 \), and \( b > 0 \). As the independent variable \( x \) changes by a constant amount, the dependent variable \( y \) is multiplied by a constant factor, which means consecutive \( y \)-values form a constant ratio.

Identifying and Evaluating Exponential Functions

Does each table represent an exponential function? Explain.

a. \[
\begin{array}{c|c|c|c|c}
 x & 0 & 1 & 2 & 3 \\
 y & 2 & 4 & 12 & 48 \\
\end{array}
\]

b. \[
\begin{array}{c|c|c|c|c}
 x & 0 & 1 & 2 & 3 \\
 y & 4 & 8 & 16 & 32 \\
\end{array}
\]

**SOLUTION**

a. As \( x \) increases by 1, \( y \) is multiplied by 2. So, the function is exponential.

b. As \( x \) increases by 1, \( y \) is not multiplied by a constant factor. So, the function is not exponential.

Evaluate each function for the given value of \( x \).

a. \( y = -2(5)^x; x = 3 \)

b. \( y = 3(0.5)^x; x = -2 \)

**SOLUTION**

a. Write the function. Substitute for \( x \). Evaluate the power. Multiply.

\[ y = -2(5)^3 = -2(125) = -250 \]

b. \( y = 3(0.5)^{-2} = 3(4) = 12 \)

Does the table represent an exponential function? Explain.

1. \[
\begin{array}{c|c|c|c|c}
 x & 0 & 1 & 2 & 3 \\
 y & 8 & 4 & 2 & 1 \\
\end{array}
\]

2. \[
\begin{array}{c|c|c|c|c}
 x & -4 & 0 & 4 & 8 \\
 y & 1 & 0 & -1 & -2 \\
\end{array}
\]

Evaluate the function when \( x = -2, 0, \) and 3.

3. \( y = 2(9)^x \)

4. \( y = 1.5(2)^x \)
Graphing Exponential Functions

The graph of a function $y = ab^x$ is a vertical stretch or shrink by a factor of $|a|$ of the graph of the parent function $y = b^x$. When $a < 0$, the graph is also reflected in the $x$-axis. The $y$-intercept of the graph of $y = ab^x$ is $a$.

**Core Concept**

**Graphing $y = ab^x$ When $b > 1$**

Graph $f(x) = 4(2)^x$. Compare the graph to the graph of the parent function. Describe the domain and range of $f$.

**SOLUTION**

Step 1 Make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$1$</td>
<td>$2$</td>
<td>$4$</td>
<td>$8$</td>
<td>$16$</td>
</tr>
</tbody>
</table>

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

The parent function is $g(x) = 2^x$. The graph of $f$ is a vertical stretch by a factor of 4 of the graph of $g$. The $y$-intercept of the graph of $f$, 4, is above the $y$-intercept of the graph of $g$, 1. From the graph of $f$, you can see that the domain is all real numbers and the range is $y > 0$.

**Graphing $y = ab^x$ When $0 < b < 1$**

Graph $f(x) = -\left(\frac{1}{2}\right)^x$. Compare the graph to the graph of the parent function. Describe the domain and range of $f$.

**SOLUTION**

Step 1 Make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$-4$</td>
<td>$-2$</td>
<td>$-1$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{4}$</td>
</tr>
</tbody>
</table>

Step 2 Plot the ordered pairs.

Step 3 Draw a smooth curve through the points.

The parent function is $g(x) = \left(\frac{1}{2}\right)^x$. The graph of $f$ is a reflection in the $x$-axis of the graph of $g$. The $y$-intercept of the graph of $f$, $-1$, is below the $y$-intercept of the graph of $g$, 1. From the graph of $f$, you can see that the domain is all real numbers and the range is $y < 0$.

**Monitoring Progress**

Graph the function. Compare the graph to the graph of the parent function. Describe the domain and range of $f$.

5. $f(x) = -2(4)^x$  
6. $f(x) = 2\left(\frac{1}{4}\right)^x$
To graph a function of the form \( y = ab^{x-h} + k \), begin by graphing \( y = ab^{x} \). Then translate the graph horizontally \( h \) units and vertically \( k \) units.

**EXAMPLE 5**  
**Graphing \( y = ab^{x-h} + k \)**

Graph \( y = 4(2)^{x-3} + 2 \). Describe the domain and range.

**SOLUTION**

**Step 1**  
Graph \( y = 4(2)^{x} \). This is the same function that is in Example 3, which passes through (0, 4) and (1, 8).

**Step 2**  
Translate the graph 3 units right and 2 units up. The graph passes through (3, 6) and (4, 10).

Notice that the graph approaches the line \( y = 2 \) but does not intersect it.

- From the graph, you can see that the domain is all real numbers and the range is \( y > 2 \).

**EXAMPLE 6**  
**Comparing Exponential Functions**

An exponential function \( g \) models a relationship in which the dependent variable is multiplied by 1.5 for every 1 unit the independent variable \( x \) increases. Graph \( g \) when \( g(0) = 4 \). Compare \( g \) and the function \( f \) from Example 3 over the interval \( x = 0 \) to \( x = 2 \).

**SOLUTION**

You know (0, 4) is on the graph of \( g \). To find points to the right of (0, 4), multiply \( g(x) \) by 1.5 for every 1 unit increase in \( x \). To find points to the left of (0, 4), divide \( g(x) \) by 1.5 for every 1 unit decrease in \( x \).

**Step 1**  
Make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>2.7</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>13.5</td>
</tr>
</tbody>
</table>

**Step 2**  
Plot the ordered pairs.

**Step 3**  
Draw a smooth curve through the points.

- Both functions have the same value when \( x = 0 \), but the value of \( f \) is greater than the value of \( g \) over the rest of the interval.

**Monitoring Progress**  
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Graph the function. Describe the domain and range.

7. \( y = -2(3)^{x+2} - 1 \)

8. \( f(x) = (0.25)^{x} + 3 \)

9. WHAT IF? In Example 6, the dependent variable of \( g \) is multiplied by 3 for every 1 unit the independent variable \( x \) increases. Graph \( g \) when \( g(0) = 4 \). Compare \( g \) and the function \( f \) from Example 3 over the interval \( x = 0 \) to \( x = 2 \).
Solving Real-Life Problems

For an exponential function of the form $y = ab^x$, the $y$-values change by a factor of $b$ as $x$ increases by 1. You can use this fact to write an exponential function when you know the $y$-intercept, $a$. The table represents the exponential function $y = 2(5)^x$.

### EXAMPLE 7  Modeling with Mathematics

The graph represents a bacterial population $y$ after $x$ days.

**a.** Write an exponential function that represents the population.

**b.** Find the population after 5 days.

#### SOLUTION

1. **Understand the Problem** You have a graph of the population that shows some data points. You are asked to write an exponential function that represents the population and find the population after a given amount of time.

2. **Make a Plan** Use the graph to make a table of values. Use the table and the $y$-intercept to write an exponential function. Then evaluate the function to find the population.

3. **Solve the Problem**

   - **a.** Use the graph to make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>3</td>
<td>12</td>
<td>48</td>
<td>192</td>
<td>768</td>
</tr>
</tbody>
</table>

   The $y$-intercept is 3. The $y$-values increase by a factor of 4 as $x$ increases by 1.

   - **So,** the population can be modeled by $y = 3(4)^x$.

   - **b.** To find the population after 5 days, evaluate the function when $x = 5$.

   $y = 3(4)^5$

   $= 3(1024)$

   $= 3072$

   Multiply.

   - **There are 3072 bacteria after 5 days.**

4. **Look Back** The graph resembles an exponential function of the form $y = ab^x$, where $b > 1$ and $a > 0$. So, the exponential function $y = 3(4)^x$ is reasonable.

#### Monitoring Progress

10. A bacterial population $y$ after $x$ days can be represented by an exponential function whose graph passes through (0, 100) and (1, 200). (a) Write a function that represents the population. (b) Find the population after 6 days. (c) Does this bacterial population grow faster than the bacterial population in Example 7? Explain.
6.1 Exercises

Vocabulary and Core Concept Check

1. OPEN-ENDED Sketch an increasing exponential function whose graph has a y-intercept of 2.
2. REASONING Why is \( a \) the y-intercept of the graph of the function \( y = ab^x \)?
3. WRITING Compare the graph of \( y = 2(5)^x \) with the graph of \( y = 5^x \).
4. WHICH ONE DOESN'T BELONG? Which equation does not belong with the other three? Explain your reasoning.

\[ y = 3^x \quad f(x) = 2(4)^x \quad f(x) = (−3)^x \quad y = 5(3)^x \]

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, determine whether the equation represents an exponential function. Explain.

5. \( y = 4(7)^x \)
6. \( y = −6x \)
7. \( y = 2x^3 \)
8. \( y = −3^x \)
9. \( y = 9(−5)^x \)
10. \( y = \frac{1}{2}(1)^x \)

In Exercises 11–14, determine whether the table represents an exponential function. Explain. (See Example 1.)

11. \[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & −2 \\
2 & −10 \\
3 & −40 \\
4 & −120 \\
\end{array}
\]

12. \[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & 6 \\
2 & 12 \\
3 & 24 \\
4 & 48 \\
\end{array}
\]

13. \[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
−1 & 0.25 \\
0 & 1 \\
1 & 4 \\
2 & 16 \\
3 & 64 \\
\end{array}
\]

14. \[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
−3 & 0 \\
0 & 3 \\
3 & 6 \\
6 & 9 \\
\end{array}
\]

15. \( y = 3^x; x = 2 \)
16. \( f(x) = 3(2)^x; x = −1 \)
17. \( y = −4(5)^x; x = 2 \)
18. \( f(x) = 0.5^x; x = −3 \)
19. \( f(x) = \frac{1}{2}(6)^x; x = 3 \)
20. \( y = \frac{1}{4}(4)^x; x = 5 \)

Using Structure In Exercises 21–24, match the function with its graph.

21. \( f(x) = 2(0.5)^x \)
22. \( y = −2(0.5)^x \)
23. \( y = 2(2)^x \)
24. \( f(x) = −2(2)^x \)

A. \( \begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & 2 \\
2 & 4 \\
3 & 8 \\
\end{array} \)

B. \( \begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & 1 \\
2 & 2 \\
3 & 4 \\
\end{array} \)

C. \( \begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & 1 \\
2 & 2 \\
3 & 4 \\
\end{array} \)

D. \( \begin{array}{c|c}
\text{x} & \text{y} \\
\hline
1 & 1 \\
2 & 2 \\
3 & 4 \\
\end{array} \)

In Exercises 25–30, graph the function. Compare the graph to the graph of the parent function. Describe the domain and range of \( f \). (See Examples 3 and 4.)

25. \( f(x) = 3(0.5)^x \)
26. \( f(x) = −4^x \)
27. \( f(x) = −2(7)^x \)
28. \( f(x) = 6\left(\frac{1}{3}\right)^x \)
29. \( f(x) = \frac{1}{2}(8)^x \)
30. \( f(x) = \frac{1}{2}(0.25)^x \)

In Exercises 31–36, graph the function. Describe the domain and range. (See Example 5.)

31. \( f(x) = 3^x − 1 \)
32. \( f(x) = 4^x + 3 \)
33. \[ y = 5^x - 2 + 7 \]
34. \[ y = -\left(\frac{1}{2}\right)^x + 1 - 3 \]
35. \[ y = -8(0.75)^x + 2 - 2 \]
36. \[ f(x) = 3(6)^x - 1 - 5 \]

In Exercises 37–40, compare the graphs. Find the value of \( h, k, \) or \( a. \)

37. \[ g(x) = a(2)^x \]
38. \[ g(x) = 0.25^x + k \]
39. \[ g(x) = -3^x - h \]
40. \[ g(x) = \frac{1}{3}(6)^x - h \]

41. ERROR ANALYSIS Describe and correct the error in evaluating the function.

\[ g(x) = 6(0.5)^x; x = -2 \]
\[ g(-2) = 6(0.5)^{-2} \]
\[ = 3^{-2} \]
\[ = \frac{1}{9} \]

42. ERROR ANALYSIS Describe and correct the error in finding the domain and range of the function.

In Exercises 43 and 44, graph the function with the given description. Compare the function to \( f(x) = 0.5(4)^x \) over the interval \( x = 0 \) to \( x = 2. \) (See Example 6.)

43. An exponential function \( g \) models a relationship in which the dependent variable is multiplied by 2.5 for every 1 unit the independent variable \( x \) increases. The value of the function at 0 is 8.

44. An exponential function \( h \) models a relationship in which the dependent variable is multiplied by \( \frac{1}{5} \) for every 1 unit the independent variable \( x \) increases. The value of the function at 0 is 32.

45. MODELING WITH MATHEMATICS You graph an exponential function on a calculator. You zoom in repeatedly to 25% of the screen size. The function \( y = 0.25^x \) represents the percent (in decimal form) of the original screen display that you see, where \( x \) is the number of times you zoom in.

a. Graph the function. Describe the domain and range.

b. Find and interpret the \( y \)-intercept.

c. You zoom in twice. What percent of the original screen do you see?

46. MODELING WITH MATHEMATICS A population \( y \) of coyotes in a national park triples every 20 years. The function \( y = 15(3)^x \) represents the population, where \( x \) is the number of 20-year periods.

a. Graph the function. Describe the domain and range.

b. Find and interpret the \( y \)-intercept.

c. How many coyotes are in the national park in 40 years?

In Exercises 47–50, write an exponential function represented by the table or graph. (See Example 7.)

47. \[
\begin{array}{cccc}
x & 0 & 1 & 2 \\
y & 2 & 14 & 98 \\
\end{array}
\]

48. \[
\begin{array}{cccc}
x & 0 & 1 & 2 \\
y & -50 & -10 & -2 \\
\end{array}
\]

49. \[
\begin{array}{cccc}
x & (1, -1) & (2, 2) & (3, -4) \\
y & (0, 0) & (1, 4) & (2, -2) \\
\end{array}
\]

50. \[
\begin{array}{cccc}
x & (1, 4) & (2, -2) & (3, 1) \\
y & (0, 0) & (1, 4) & (2, -2) \\
\end{array}
\]
51. **MODELING WITH MATHEMATICS** The graph represents the number \(y\) of visitors to a new art gallery after \(x\) months.

<table>
<thead>
<tr>
<th>Month</th>
<th>Visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0, 40)</td>
</tr>
<tr>
<td>1</td>
<td>(1, 60)</td>
</tr>
<tr>
<td>2</td>
<td>(2, 90)</td>
</tr>
<tr>
<td>3</td>
<td>(3, 135)</td>
</tr>
</tbody>
</table>

a. Write an exponential function that represents this situation.

b. Approximate the number of visitors after 5 months.

52. **PROBLEM SOLVING** A sales report shows that 3300 gas grills were purchased from a chain of hardware stores last year. The store expects grill sales to increase 6% each year. About how many grills does the store expect to sell in Year 6? Use an equation to justify your answer.

53. **WRITING** Graph the function \(f(x) = -2^x\). Then graph \(g(x) = -2^x - 3\). How are the \(y\)-intercept, domain, and range affected by the translation?

54. **MAKING AN ARGUMENT** Your friend says that the table represents an exponential function because \(y\) is multiplied by a constant factor. Is your friend correct? Explain.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>2</td>
<td>10</td>
<td>50</td>
<td>250</td>
</tr>
</tbody>
</table>

55. **WRITING** Describe the effect of \(a\) on the graph of \(y = a \cdot 2^x\) when \(a\) is positive and when \(a\) is negative.

56. **OPEN-ENDED** Write a function whose graph is a horizontal translation of the graph of \(h(x) = 4^x\).

57. **USING STRUCTURE** The graph of \(g\) is a translation 4 units up and 3 units right of the graph of \(f(x) = 5^x\). Write an equation for \(g\).

58. **HOW DO YOU SEE IT?** The exponential function \(y = V(x)\) represents the projected value of a stock \(x\) weeks after a corporation loses an important legal battle. The graph of the function is shown.

a. After how many weeks will the stock be worth $20?

b. Describe the change in the stock price from Week 1 to Week 3.

59. **USING GRAPHS** The graph represents the exponential function \(f\). Find \(f(7)\).

60. **THOUGHT PROVOKING** Write a function of the form \(y = ab^x\) that represents a real-life population. Explain the meaning of each of the constants \(a\) and \(b\) in the real-life context.

61. **REASONING** Let \(f(x) = ab^x\). Show that when \(x\) is increased by a constant \(k\), the quotient \(\frac{f(x + k)}{f(x)}\) is always the same regardless of the value of \(x\).

62. **PROBLEM SOLVING** A function \(g\) models a relationship in which the dependent variable is multiplied by 4 for every 2 units the independent variable increases. The value of the function at 0 is 5. Write an equation that represents the function.

63. **PROBLEM SOLVING** Write an exponential function \(f\) so that the slope from the point \((0, f(0))\) to the point \((2, f(2))\) is equal to 12.

### Maintaining Mathematical Proficiency

Write the percent as a decimal. *(Skills Review Handbook)*

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>64.</td>
<td>4%</td>
</tr>
<tr>
<td>65.</td>
<td>35%</td>
</tr>
<tr>
<td>66.</td>
<td>128%</td>
</tr>
<tr>
<td>67.</td>
<td>250%</td>
</tr>
</tbody>
</table>

280 Chapter 6 Exponential Functions and Sequences
6.2 Exponential Growth and Decay

Essential Question What are some of the characteristics of exponential growth and exponential decay functions?

Exploration 1 Predicting a Future Event

Work with a partner. It is estimated, that in 1782, there were about 100,000 nesting pairs of bald eagles in the United States. By the 1960s, this number had dropped to about 500 nesting pairs. In 1967, the bald eagle was declared an endangered species in the United States. With protection, the nesting pair population began to increase. Finally, in 2007, the bald eagle was removed from the list of endangered and threatened species.

Describe the pattern shown in the graph. Is it exponential growth? Assume the pattern continues. When will the population return to that of the late 1700s? Explain your reasoning.

Exploration 2 Describing a Decay Pattern

Work with a partner. A forensic pathologist was called to estimate the time of death of a person. At midnight, the body temperature was 80.5°F and the room temperature was a constant 60°F. One hour later, the body temperature was 78.5°F.

a. By what percent did the difference between the body temperature and the room temperature drop during the hour?

b. Assume that the original body temperature was 98.6°F. Use the percent decrease found in part (a) to make a table showing the decreases in body temperature. Use the table to estimate the time of death.

Communicate Your Answer

3. What are some of the characteristics of exponential growth and exponential decay functions?

4. Use the Internet or some other reference to find an example of each type of function. Your examples should be different than those given in Explorations 1 and 2.
   a. exponential growth
   b. exponential decay
What You Will Learn

- Use and identify exponential growth and decay functions.
- Solve real-life problems involving exponential growth and decay.

Exponential Growth and Decay Functions

Exponential growth occurs when a quantity increases by the same factor over equal intervals of time.

Core Concept

Exponential Growth Functions

A function of the form \( y = a(1 + r)^t \), where \( a > 0 \) and \( r > 0 \), is an exponential growth function.

STUDY TIP

Notice that an exponential growth function is of the form \( y = ab^x \), where \( b \) is replaced by \( 1 + r \) and \( x \) is replaced by \( t \).

EXAMPLE 1

Using an Exponential Growth Function

The inaugural attendance of an annual music festival is 150,000. The attendance \( y \) increases by 8% each year.

a. Write an exponential growth function that represents the attendance after \( t \) years.

b. How many people will attend the festival in the fifth year? Round your answer to the nearest thousand.

SOLUTION

a. The initial amount is 150,000, and the rate of growth is 8%, or 0.08.

\[
y = a(1 + r)^t = 150,000(1 + 0.08)^t = 150,000(1.08)^t
\]

The festival attendance can be represented by \( y = 150,000(1.08)^t \).

b. The value \( t = 4 \) represents the fifth year because \( t = 0 \) represents the first year.

\[
y = 150,000(1.08)^4 = 150,000(1.08)^4 \approx 204,073
\]

About 204,000 people will attend the festival in the fifth year.

Monitoring Progress

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1. A website has 500,000 members in 2010. The number \( y \) of members increases by 15% each year. (a) Write an exponential growth function that represents the website membership \( t \) years after 2010. (b) How many members will there be in 2016? Round your answer to the nearest ten thousand.
Exponential decay occurs when a quantity decreases by the same factor over equal intervals of time.

**Core Concept**

**Exponential Decay Functions**

A function of the form $y = a(1 - r)^t$, where $a > 0$ and $0 < r < 1$, is an exponential decay function.

**STUDY TIP**

Notice that an exponential decay function is of the form $y = ab^x$, where $b$ is replaced by $1 - r$ and $x$ is replaced by $t$.

For exponential growth, the value inside the parentheses is greater than 1 because $r$ is added to 1. For exponential decay, the value inside the parentheses is less than 1 because $r$ is subtracted from 1.

**Example 2** Identifying Exponential Growth and Decay

Determine whether each table represents an exponential growth function, an exponential decay function, or neither.

a. | $x$ | $y$ |
---|---|---|
| 0  | 270 |
| 1  | 90  |
| 2  | 30  |
| 3  | 10  |

**SOLUTION**

a. As $x$ increases by 1, $y$ is multiplied by $\frac{1}{3}$. So, the table represents an exponential decay function.

b. | $x$ | 0 | 1 | 2 | 3 |
---|---|---|---|---|
| $y$ | 5 | 10 | 20 | 40 |

b. As $x$ increases by 1, $y$ is multiplied by 2. So, the table represents an exponential growth function.

**Monitoring Progress**

Determine whether the table represents an exponential growth function, an exponential decay function, or neither. Explain.

2. | $x$ | 0 | 1 | 2 | 3 |
---|---|---|---|---|
| $y$ | 64 | 16 | 4 | 1 |

3. | $x$ | 1 | 3 | 5 | 7 |
---|---|---|---|---|
| $y$ | 4 | 11 | 18 | 25 |
**Example 3**  
Interpreting Exponential Functions

Determine whether each function represents exponential growth or exponential decay. Identify the percent rate of change.

a. \( y = 5(1.07)^t \)

b. \( f(t) = 0.2(0.98)^t \)

**Solution**

a. The function is of the form \( y = a(1 + r)^t \), where \( 1 + r > 1 \), so it represents exponential growth. Use the growth factor \( 1 + r \) to find the rate of growth.

\[
1 + r = 1.07 \quad \text{Write an equation.} \\
\]

\[
r = 0.07 \quad \text{Solve for } r. \\
\]

\[\text{So, the function represents exponential growth and the rate of growth is 7%}.\]

b. The function is of the form \( y = a(1 - r)^t \), where \( 1 - r < 1 \), so it represents exponential decay. Use the decay factor \( 1 - r \) to find the rate of decay.

\[
1 - r = 0.98 \quad \text{Write an equation.} \\
\]

\[
r = 0.02 \quad \text{Solve for } r. \\
\]

\[\text{So, the function represents exponential decay and the rate of decay is 2%}.\]

**Monitoring Progress**

Help in English and Spanish at BigIdeasMath.com

Determine whether the function represents exponential growth or exponential decay. Identify the percent rate of change.

4. \( y = 2(0.92)^t \)

5. \( f(t) = (1.2)^t \)

**Solving Real-Life Problems**

Exponential growth functions are used in real-life situations involving compound interest. Although interest earned is expressed as an annual rate, the interest is usually compounded more frequently than once per year. So, the formula \( y = a(1 + r)^t \) must be modified for compound interest problems.

**Study Tip**

For interest compounded yearly, you can substitute 1 for \( n \) in the formula to get \( y = P(1 + r)^t \).

**Core Concept**

**Compound Interest**

Compound interest is the interest earned on the principal and on previously earned interest. The balance \( y \) of an account earning compound interest is

\[
y = P \left(1 + \frac{r}{n}\right)^{nt}. \\
\]

- \( P \) = principal (initial amount)
- \( r \) = annual interest rate (in decimal form)
- \( t \) = time (in years)
- \( n \) = number of times interest is compounded per year
**EXAMPLE 4**

**Writing a Function**

You deposit $100 in a savings account that earns 6% annual interest compounded monthly. Write a function that represents the balance after $t$ years.

**SOLUTION**

\[ y = P \left( 1 + \frac{r}{n} \right)^n \]

Write the compound interest formula.

\[ = 100 \left( 1 + \frac{0.06}{12} \right)^{12t} \]

Substitute 100 for $P$, 0.06 for $r$, and 12 for $n$.

\[ = 100(1.005)^{12t} \]

Simplify.

**EXAMPLE 5**

**Solving a Real-Life Problem**

You have a checking account and a money market account at a local bank. Your checking account has a constant balance of $200. The table shows the total balance of the accounts over time.

a. Write a function $m$ that represents the balance of your money market account after $t$ years.

b. Write a function $B$ that represents the total balance after $t$ years. Compare the graph of $m$ to the graph of $B$.

c. Compare the money market account to the savings account in Example 4.

**SOLUTION**

a. The $200 balance of your checking account can be represented by the constant function $c(t) = 200$. To find the balances $m(t)$ of the money market account, subtract $200 from each value in the table, as shown.

From this table, you know the initial balance is $100, and it increases 10% each year. So, $P = 100$ and $r = 0.1$.

\[ m(t) = P(1 + r)^t \]

Write the compound interest formula when $n = 1$.

\[ = 100(1 + 0.1)^t \]

Substitute 100 for $P$ and 0.1 for $r$.

\[ = 100(1.1)^t \]

Add.

b. To write a function that represents the total balance, find the sum of the expressions that represent the balances of the two accounts.

\[ B(t) = m(t) + c(t) \]

\[ = 100(1.1)^t + 200 \]

From the graphs, you can see that the graph of $B$ is a vertical translation of the graph of $m$.

c. Each account has an initial balance of $100. The money market account earns 10% interest each year, and the savings account earns 6% interest each year. So, the balance of the money market account increases faster.

**Monitoring Progress**

6. You deposit $500 in an account that earns 9% annual interest compounded monthly. Write a function that represents the balance $y$ (in dollars) after $t$ years.

7. WHAT IF? Repeat Example 5 using the table shown.
6.2 Exercises

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** In the exponential growth function \( y = a(1 + r)^t \), the quantity \( r \) is called the ________.

2. **VOCABULARY** What is the decay factor in the exponential decay function \( y = a(1 - r)^t \)?

3. **VOCABULARY** Compare exponential growth and exponential decay.

4. **WRITING** When does the function \( y = ab^t \) represent exponential growth? exponential decay?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, identify the initial amount \( a \) and the rate of growth \( r \) (as a percent) of the exponential function. Evaluate the function when \( t = 5 \). Round your answer to the nearest tenth.

5. \( y = 350(1 + 0.75)^t \)  

6. \( y = 10(1 + 0.4)^t \)  

7. \( y = 25(1.2)^t \)  

8. \( y = 12(1.05)^t \)  

9. \( f(t) = 1500(1.074)^t \)  

10. \( h(t) = 175(1.028)^t \)  

11. \( g(t) = 6.72(2)^t \)  

12. \( p(t) = 1.8^t \)

In Exercises 13–16, write a function that represents the situation.

13. Sales of $10,000 increase by 65% each year.

14. Your starting annual salary of $35,000 increases by 4% each year.

15. A population of 210,000 increases by 12.5% each year.

16. An item costs $4.50, and its price increases by 3.5% each year.

17. **MODELING WITH MATHEMATICS** The population of a city has been increasing by 2% annually. The sign shown is from the year 2000. (See Example 1.)

   a. Write an exponential growth function that represents the population \( t \) years after 2000.

   b. What will the population be in 2020? Round your answer to the nearest thousand.

18. **MODELING WITH MATHEMATICS** A young channel catfish weighs about 0.1 pound. During the next 8 weeks, its weight increases by about 23% each week.

   a. Write an exponential growth function that represents the weight of the catfish after \( t \) weeks during the 8-week period.

   b. About how much will the catfish weigh after 4 weeks? Round your answer to the nearest thousandth.

In Exercises 19–26, identify the initial amount \( a \) and the rate of decay \( r \) (as a percent) of the exponential function. Evaluate the function when \( t = 3 \). Round your answer to the nearest tenth.

19. \( y = 575(1 - 0.6)^t \)  

20. \( y = 8(1 - 0.15)^t \)  

21. \( g(t) = 240(0.75)^t \)  

22. \( f(t) = 475(0.5)^t \)  

23. \( w(t) = 700(0.995)^t \)  

24. \( h(t) = 1250(0.865)^t \)  

25. \( y = \left(\frac{2}{5}\right)^t \)  

26. \( y = 0.5\left(\frac{3}{4}\right)^t \)

In Exercises 27–29, write a function that represents the situation.

27. A population of 100,000 decreases by 2% each year.

28. A $900 sound system decreases in value by 9% each year.

29. A stock valued at $100 decreases in value by 9.5% each year.
30. **ERROR ANALYSIS** You purchase a car in 2010 for $25,000. The value of the car decreases by 14% annually. Describe and correct the error in finding the value of the car in 2015.

\[
v(t) = 25,000(1.14)^t\\
v(5) = 25,000(1.14)^5 \approx 48,135\\
The value of the car in 2015 is about $48,000.
\]

In Exercises 31–36, determine whether the table represents an exponential growth function, an exponential decay function, or neither. Explain.
(See Example 2.)

<table>
<thead>
<tr>
<th>31. x</th>
<th>y</th>
<th>32. x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>50</td>
<td>0</td>
<td>32</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>1</td>
<td>28</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>3</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>33. x</th>
<th>y</th>
<th>34. x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>35</td>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>1</td>
<td>29</td>
<td>2</td>
<td>51</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>3</td>
<td>153</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>4</td>
<td>459</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>35. x</th>
<th>y</th>
<th>36. x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>432</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>5</td>
<td>72</td>
</tr>
<tr>
<td>15</td>
<td>32</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>20</td>
<td>128</td>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

37. **ANALYZING RELATIONSHIPS** The table shows the value of a camper \( t \) years after it is purchased.

- a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.
- b. What is the value of the camper after 5 years?

<table>
<thead>
<tr>
<th>( t )</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$37,000</td>
</tr>
<tr>
<td>2</td>
<td>$29,600</td>
</tr>
<tr>
<td>3</td>
<td>$23,680</td>
</tr>
<tr>
<td>4</td>
<td>$18,944</td>
</tr>
</tbody>
</table>

38. **ANALYZING RELATIONSHIPS** The table shows the total numbers of visitors to a website \( t \) days after it is online.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>11,000</td>
</tr>
<tr>
<td>43</td>
<td>12,100</td>
</tr>
<tr>
<td>44</td>
<td>13,310</td>
</tr>
<tr>
<td>45</td>
<td>14,641</td>
</tr>
</tbody>
</table>

- a. Determine whether the table represents an exponential growth function, an exponential decay function, or neither.
- b. How many people will have visited the website after it is online 47 days?

In Exercises 39–46, determine whether each function represents exponential growth or exponential decay. Identify the percent rate of change.
(See Example 3.)

<table>
<thead>
<tr>
<th>39. ( y = 4(0.8)^t )</th>
<th>40. ( y = 15(1.1)^t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>41. ( y = 30(0.95)^t )</td>
<td>42. ( y = 5(1.08)^t )</td>
</tr>
<tr>
<td>43. ( r(t) = 0.4(1.06)^t )</td>
<td>44. ( s(t) = 0.65(0.48)^t )</td>
</tr>
<tr>
<td>45. ( g(t) = 2\left(\frac{5}{4}\right)^t )</td>
<td>46. ( m(t) = \left(\frac{4}{5}\right)^t )</td>
</tr>
</tbody>
</table>

In Exercises 47–50, write a function that represents the balance after \( t \) years.
(See Example 4.)

- 47. $2000 deposit that earns 5% annual interest compounded quarterly
- 48. $1400 deposit that earns 10% annual interest compounded semiannually
- 49. $6200 deposit that earns 8.4% annual interest compounded monthly
- 50. $3500 deposit that earns 9.2% annual interest compounded quarterly

51. **PROBLEM SOLVING** You have a checking account and a savings account at a credit union. Your checking account has a constant balance of $500. The table shows the total balance of the accounts over time.
(See Example 5.)

- a. Write a function \( m \) that represents the balance of your savings account after \( t \) years.
- b. Write a function \( B \) that represents the total balance after \( t \) years. Compare the graph of \( m \) to the graph of \( B \).
- c. Compare the savings account to the account represented in Exercise 47.

<table>
<thead>
<tr>
<th>Year, ( t )</th>
<th>Total balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2500</td>
</tr>
<tr>
<td>1</td>
<td>$2540</td>
</tr>
<tr>
<td>2</td>
<td>$2580.80</td>
</tr>
<tr>
<td>3</td>
<td>$2622.42</td>
</tr>
<tr>
<td>4</td>
<td>$2664.86</td>
</tr>
<tr>
<td>5</td>
<td>$2708.16</td>
</tr>
</tbody>
</table>
52. **COMBINING FUNCTIONS** You deposit $9000 in a savings account that earns 3.6% annual interest compounded monthly. You also save $40 per month in a safe at home. Write a function \( C(t) = b(t) + h(t) \), where \( b(t) \) represents the balance of your savings account and \( h(t) \) represents the amount in your safe after \( t \) years. Find and interpret \( C(5) \).

53. **NUMBER SENSE** During a flu epidemic, the number of sick people triples every week. What is the growth rate as a percent? Explain your reasoning.

54. **HOW DO YOU SEE IT?** Match each situation with its graph. Explain your reasoning.
   a. A bacterial population doubles each hour.
   b. The value of a computer decreases by 18% each year.
   c. A deposit earns 11% annual interest compounded yearly.
   d. A radioactive element decays 5.5% each year.

55. **WRITING** Give an example of an equation in the form \( y = ab^x \) that does not represent an exponential growth function or an exponential decay function. Explain your reasoning.

56. **THOUGHT PROVOKING** Describe two account options into which you can deposit $1000 and earn compound interest. Write a function that represents the balance of each account after \( t \) years. Which account would you rather use? Explain your reasoning.

57. **MAKING AN ARGUMENT** A store is having a sale on sweaters. On the first day, the prices of the sweaters are reduced by 20%. The prices will be reduced another 20% each day until the sweaters are sold. Your friend says the sweaters will be free on the fifth day. Is your friend correct? Explain.

58. **COMPARING FUNCTIONS** The graphs of \( f \) and \( g \) are shown.

   a. Explain why \( f \) is an exponential growth function. Identify the rate of growth.
   b. Describe the transformation from the graph of \( f \) to the graph of \( g \). Determine the value of \( k \).
   c. The graph of \( g \) is the same as the graph of \( h(t) = f(t + r) \). Find the value of \( r \). Explain your procedure.

### Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The points represented by the table lie on a line. Find the slope of the line.  
\[
\begin{array}{c|c|c|c|c}
59. & x & -1 & 1 & 3 & 5 \\
   y & 10 & 5 & 0 & -5 \\
\end{array}
\]

Write an equation of the line that passes through the given points.  
\[
\begin{array}{c|c|c|c|c}
60. & x & -6 & -3 & 0 & 3 \\
   y & -2 & 2 & 6 & 10 \\
\end{array}
\]

61. \((0, -2), (3, -5)\)  
62. \((-4, -2), (0, 6)\)  
63. \((-7, -4), (-1, 5)\)
**6.3 Comparing Linear and Exponential Functions**

**Essential Question** How can you compare the growth rates of linear and exponential functions?

**EXPLORATION 1** Comparing Values

Work with a partner. An art collector buys two paintings. The value of each painting after \( t \) years is \( y \) dollars. Complete each table. Compare the values of the two paintings. Which painting’s value has a constant growth rate? Which painting’s value has an increasing growth rate? Explain your reasoning.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y = 19t + 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>1</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>2</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>3</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>4</td>
<td>[\text{value}]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y = 3^t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>1</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>2</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>3</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>4</td>
<td>[\text{value}]</td>
</tr>
</tbody>
</table>

**EXPLORATION 2** Comparing Values

Work with a partner. Analyze the values of the two paintings over the given time periods. The value of each painting after \( t \) years is \( y \) dollars. Which painting’s value eventually overtakes the other?

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y = 19t + 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>5</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>6</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>7</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>8</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>9</td>
<td>[\text{value}]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y = 3^t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>5</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>6</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>7</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>8</td>
<td>[\text{value}]</td>
</tr>
<tr>
<td>9</td>
<td>[\text{value}]</td>
</tr>
</tbody>
</table>

**EXPLORATION 3** Comparing Graphs

Work with a partner. Use the tables in Explorations 1 and 2 to graph \( y = 19t + 5 \) and \( y = 3^t \) in the same coordinate plane. Compare the graphs of the functions.

**Communicate Your Answer**

4. How can you compare the growth rates of linear and exponential functions?

5. Which function has a growth rate that is eventually much greater than the growth rate of the other function? Explain your reasoning.
What You Will Learn

- Choose functions to model data.
- Compare functions using average rates of change.
- Solve real-life problems involving different function types.

Choosing Functions to Model Data

So far, you have studied linear functions and exponential functions. You can use these functions to model data.

Core Concept

Linear and Exponential Functions

Linear Function

\[ y = mx + b \]

Exponential Function

\[ y = ab^x \]

EXAMPLE 1 Using Graphs to Identify Functions

Plot the points. Tell whether the points appear to represent a linear function, an exponential function, or neither.

a. \((3, -1), (2, 1), (0, 1), (1, 3), (-1, -1)\)

b. \((0, 1), (2, -2), (4, -5), (-2, 4), (-4, 7)\)

c. \((0, 2), (1, 1), \left(2, \frac{1}{2}\right), (-2, 8), (-1, 4)\)

SOLUTION

a. neither

b. linear

c. exponential

Monitoring Progress

Plot the points. Tell whether the points appear to represent a linear function, an exponential function, or neither.

1. \((-3, -4), (0, 2), (-2, 2), (1, -4), (-1, 4)\)

2. \((1, 1), (2, 3), (3, 9), \left(0, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\)

3. \((-2, -7), (0, -1), (2, 5), (-1, -4), (1, 2)\)

Chapter 6 Exponential Functions and Sequences
Core Concept

Differences and Ratios of Functions
You can use patterns between consecutive data pairs to determine which type of function models the data.

• **Linear Function** The differences of consecutive y-values are constant.
• **Exponential Function** Consecutive y-values have a common ratio.

In each case, the differences of consecutive x-values need to be constant.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

Tell whether each table of values represents a **linear** or an **exponential** function. Then write the function.

<table>
<thead>
<tr>
<th>a.</th>
<th>x</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>y</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>−3</td>
<td>−7</td>
</tr>
<tr>
<td>b.</td>
<td>x</td>
<td>−3</td>
<td>−2</td>
<td>−1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td></td>
</tr>
</tbody>
</table>

**SOLUTION**

a. 

The differences of consecutive y-values are constant. The slope is $\frac{-4}{1} = -4$ and the y-intercept is 1. So, the table represents the linear function $y = -4x + 1$.

b. 

Consecutive y-values have a common ratio of 3 and the y-intercept is 9. So, the table represents the exponential function $y = 9(3)^x$.

Monitoring Progress

4. Tell whether the table of values represents a **linear** or an **exponential** function. Then write the function.

<table>
<thead>
<tr>
<th>x</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Comparing Functions Using Average Rates of Change

For nonlinear functions, the rate of change is not constant. You can compare two functions over the same interval using their **average rate of change**. The average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Section 6.3 Comparing Linear and Exponential Functions 291
Chapter 6  Exponential Functions and Sequences

Using and Interpreting Average Rates of Change

Two social media websites open their memberships to the public. (a) Compare the websites by calculating and interpreting the average rates of change from Day 10 to Day 20. (b) Predict which website will have more members after 50 days. Explain.

### Website A

<table>
<thead>
<tr>
<th>Day, $x$</th>
<th>Members, $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>650</td>
</tr>
<tr>
<td>5</td>
<td>1025</td>
</tr>
<tr>
<td>10</td>
<td>1400</td>
</tr>
<tr>
<td>15</td>
<td>1775</td>
</tr>
<tr>
<td>20</td>
<td>2150</td>
</tr>
<tr>
<td>25</td>
<td>2525</td>
</tr>
</tbody>
</table>

### Website B

<table>
<thead>
<tr>
<th>$x$</th>
<th>Members</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>950</td>
</tr>
<tr>
<td>20</td>
<td>1850</td>
</tr>
</tbody>
</table>

**SOLUTION**

**a.** Calculate the average rates of change by using the points whose $x$-coordinates are 10 and 20.

Website A: Use (10, 1400) and (20, 2150).

\[
\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75
\]

Website B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 950) and (20, 1850).

\[
\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{1850 - 950}{20 - 10} = \frac{900}{10} = 90
\]

From Day 10 to Day 20, Website A membership increases at an average rate of 75 people per day, and Website B membership increases at an average rate of about 90 people per day. So, Website B membership is growing faster.

**b.** Using the table, membership increases and the average rates of change are constant. So, Website A membership can be represented by an increasing linear function. Using the graph, membership increases and the average rates of change are increasing. It appears that Website B membership can be represented by an increasing exponential function.

After 25 days, the memberships of both websites are about equal and the average rate of change of Website B exceeds the average rate of change of Website A. So, Website B will have more members after 50 days.

**Monitoring Progress**

5. Compare the websites in Example 3 by calculating and interpreting the average rates of change from Day 0 to Day 10.
Section 6.3
Comparing Linear and Exponential Functions

Solving Real-Life Problems

In 2000, Littleton had a population of 11,510 people. Littleton’s population increased by 600 people each year. In 2000, Tinyville had a population of 10,000 people. Tinyville’s population increased by 10% each year.

a. In what year were the populations equal?

b. Suppose Littleton’s initial population doubled to 23,020 and maintained a constant rate of increase of 600 people each year. Did Tinyville’s population still catch up to Littleton’s population?

c. Suppose Littleton’s rate of increase doubled to 1200 people each year, in addition to doubling the initial population. Did Tinyville’s population still catch up to Littleton’s population? Explain.

**SOLUTION**

a. Let $x$ represent the number of years since 2000. Write a function to model the population of each town.

Littleton: $L(x) = 600x + 11,510$  \hspace{1cm} Linear function

Tinyville: $T(x) = 10,000(1.1)^x$  \hspace{1cm} Exponential function

Use a graphing calculator to graph each function in the same viewing window. Use the **intersect** feature to find the value of $x$ for which $L(x) = T(x)$. The graphs intersect when $x = 3$.

So, the populations were equal in 2003.

b. Littleton’s new population function is $f(x) = 600x + 23,020$. Use a graphing calculator to graph $f$ and $T$ in the same viewing window.

From the graph, you can see that Tinyville’s population eventually caught up to and exceeded Littleton’s population.

c. Littleton’s new population function is $g(x) = 1200x + 23,020$. Use a graphing calculator to graph $g$ and $T$ in the same viewing window.

From the graph, you can see that Tinyville’s population eventually caught up to and exceeded Littleton’s population. Because Littleton’s population shows linear growth and Tinyville’s population shows exponential growth, Tinyville’s population eventually exceeded Littleton’s regardless of Littleton’s constant rate or initial value.

**Monitoring Progress**

6. **WHAT IF?** In 2000, Littleton had a population of 10,900 people and the population increased by 600 people each year. In what year were the populations equal?
6.3 Exercises

Vocabulary and Core Concept Check

1. **WRITING** Name two types of functions that you can use to model data. Describe the equation and graph of each type of function.

2. **WRITING** How can you decide whether to use a linear or an exponential function to model a data set?

3. **VOCABULARY** Describe how to find the average rate of change of a function \( y = f(x) \) between \( x = a \) and \( x = b \).

4. **DIFFERENT WORDS, SAME QUESTION** Let \( f(x) = 4^x \). Which is different? Find “both” answers.

   - As the input increases by 1, by what factor does the output increase?
   - What is the average rate of change of \( f \) between \( x = 1 \) and \( x = 2 \)?
   - What is \( \frac{f(2) - f(1)}{2 - 1} \)?
   - What is the slope of the line that passes through (1, 4) and (2, 16)?

Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a linear function, an exponential function, or neither.

5. ![Graph](image1)

6. ![Graph](image2)

7. ![Graph](image3)

8. ![Graph](image4)

In Exercises 9–14, plot the points. Tell whether the points appear to represent a linear function, an exponential function, or neither. (See Example 1.)

9. \((-1, -3), (0, -2), (1, -1), (2, 0), (3, 1)\)

10. \((-\frac{1}{2}, 1), (0, \frac{1}{2}), (1, 1), (2, 2), (3, 4)\)

11. \((0, 0), (4, 6), (2, 3), (-2, 3), (-4, 6)\)

12. \((-1, -1), (-2, -6), (0, -2), (3, 1), (2, -2)\)

13. \((-3, 5.5), (-1, 0.5), (0, -2), (1, -4.5), (3, -9.5)\)

14. \((0, 6), (1, 3), (2, 1.5), (3, 0.75), (-1, 12)\)

In Exercises 15–18, tell whether the table of values represents a linear or an exponential function. Then write the function. (See Example 2.)

15. | \(x\) | -1 | 0 | 1 | 2 | 3 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-1</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

16. | \(x\) | -2 | -1 | 0 | 1 | 2 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>(\frac{1}{3})</td>
<td>1</td>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>

17. | \(x\) | 1 | 2 | 3 | 4 | 5 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>512</td>
<td>128</td>
<td>32</td>
<td>8</td>
</tr>
</tbody>
</table>

18. | \(x\) | -5 | -4 | -3 | -2 | -1 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>12</td>
<td>9</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>
19. **MODELING WITH MATHEMATICS** You ride a bus to school. The table shows the distances \( d \) (in miles) the bus travels in \( t \) minutes. Let the time \( t \) represent the independent variable. Tell whether the data can be modeled by a linear function, an exponential function, or neither. Explain.

<table>
<thead>
<tr>
<th>Time, ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance, ( d )</td>
<td>0.7</td>
<td>1.4</td>
<td>2.1</td>
<td>2.1</td>
<td>2.8</td>
</tr>
</tbody>
</table>

20. **MODELING WITH MATHEMATICS** The table shows the size \( s \) (in hectares) of a glacier \( d \) decades after 1980.

<table>
<thead>
<tr>
<th>Decades, ( d )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size, ( s )</td>
<td>300</td>
<td>270</td>
<td>243</td>
<td>218.7</td>
</tr>
</tbody>
</table>

a. Plot the points. Let the number \( d \) of decades after 1980 represent the independent variable.

b. Tell whether the data can be modeled by a linear or an exponential function. Then write the function.

21. **ERROR ANALYSIS** Describe and correct the error in determining which type of function the table represents.

```
x  y
0 3
1 6
2 12
3 24
4 48

Consecutive y-values change by a constant amount. So, the table represents a linear function.
```

22. **ANALYZING RELATIONSHIPS** The population of Town A in 1970 was 3000. The population of Town A increased by 20% every decade. Let \( x \) represent the number of decades since 1970. The graph shows the population of Town B. (See Example 3.)

a. Compare the populations of the towns by calculating and interpreting the average rates of change from 1990 to 2010.

b. Predict which town will have a greater population after 2020. Explain.

23. **ANALYZING RELATIONSHIPS** Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The table shows the numbers of donations collected by Organization B. The graph shows the numbers of donations collected by Organization C.

```
<table>
<thead>
<tr>
<th>Time (hours), ( t )</th>
<th>Number of donations, ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
</tr>
</tbody>
</table>
```

a. What type of function represents the numbers of donations collected by Organization A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from \( t = 0 \) to \( t = 6 \).

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

24. **COMPARING FUNCTIONS** The room expenses for two different resorts are shown. (See Example 4.)

```
<table>
<thead>
<tr>
<th>Resort</th>
<th>Vacations available up to</th>
<th>Nightly price (including tax)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue Water Resort</td>
<td>14 nights</td>
<td>$1450 + 10% of $1450 for each additional night</td>
</tr>
<tr>
<td>Sea Breeze Resort</td>
<td>14 nights</td>
<td>$1120 + 10% of $1120 for each additional night</td>
</tr>
</tbody>
</table>
```

a. For what length of vacation does each resort cost about the same?

b. Suppose Blue Water Resort charges $1450 for the first three nights and $105 for each additional night. Would Sea Breeze Resort ever be more expensive than Blue Water Resort? Explain.

c. Suppose Sea Breeze Resort charges $1200 for the first three nights. The charge increases 10% for each additional night. Would Blue Water Resort ever be more expensive than Sea Breeze Resort? Explain.
25. **REASONING** Explain why the average rate of change of a linear function is constant and the average rate of change of an exponential function is not constant.

26. **HOW DO YOU SEE IT?** Match each graph with its function. Explain your reasoning.

   a. ![Graph a](image)
   b. ![Graph b](image)
   c. ![Graph c](image)
   d. ![Graph d](image)

   a. \( y = -\frac{1}{2}x + 1 \)
   b. \( y = 2(4)^x + 1 \)
   c. \( y = 2\left(\frac{1}{2}\right)^x + 1 \)
   d. \( y = 2x - 4 \)

27. **USING STRUCTURE** In the ordered pairs below, \( m \) is an integer and \( n \neq 0 \). Tell whether the ordered pairs represent a linear or an exponential function. Explain.

   \((m, n), (m + 1, 2n), (m + 2, 4n), (m + 3, 8n), (m + 4, 16n)\)

28. **CRITICAL THINKING** Write and graph a linear function \( f \) and an exponential function \( g \) with the following characteristics.

   - \( f(x) > g(x) \) when \( 0 < x < 4 \)
   - \( f(x) < g(x) \) when \( x < 0 \) and \( x > 4 \)

29. **CRITICAL THINKING** Is the graph of a set of points enough to determine whether the points represent a linear function or an exponential function? Justify your answer.

30. **THOUGHT PROVOKING** Find four different patterns in the figure. Determine whether each pattern represents a linear or an exponential function. Write a model for each pattern.

31. **MAKING AN ARGUMENT** Function \( p \) is an exponential function and function \( q \) is a linear function. Your friend says that after \( x = 2 \), function \( q \) will always have a greater \( y \)-value than function \( p \). Is your friend correct? Explain.

32. **USING TOOLS** The table shows the amount \( a \) (in milligrams) of venom remaining in an animal’s body \( x \) days after being bitten by a venomous snake. Let the number \( x \) of days represent the independent variable. Using technology, find a function that models the data. How did you choose the model? About how much venom was initially injected into the animal’s body? Explain your reasoning.

<table>
<thead>
<tr>
<th>Day, ( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount, ( a )</td>
<td>10.2</td>
<td>5.3</td>
<td>2.5</td>
<td>1.2</td>
<td>0.7</td>
</tr>
</tbody>
</table>

33. Solve the equation. Check your solution. (Section 1.3)

   33. \( 8x + 12 = 4x \)
   34. \( 5 - t = 7t + 21 \)
   35. \( 6(r - 2) = 2r + 8 \)
   36. \( -6(x + 7) = 10(3 - x) \)

37. Find the slope and the \( y \)-intercept of the graph of the linear equation. (Section 3.5)

   37. \( y = -6x + 7 \)
   38. \( y = \frac{1}{4}x + 7 \)
   39. \( 3y = 6x - 12 \)
   40. \( 2y + x = 8 \)
6.1–6.3 What Did You Learn?

Core Vocabulary

exponential function, p. 274
exponential growth, p. 282
exponential growth function, p. 282
exponential decay, p. 283
exponential decay function, p. 283
compound interest, p. 284
average rate of change, p. 291

Core Concepts

Section 6.1
Graphing $y = ab^x$ When $b > 1$, p. 275
Graphing $y = ab^x$ When $0 < b < 1$, p. 275

Section 6.2
Exponential Growth Functions, p. 282
Exponential Decay Functions, p. 283
Compound Interest, p. 284

Section 6.3
Linear and Exponential Functions, p. 290
Differences and Ratios of Functions, p. 291
Comparing Functions Using Average Rates of Change, p. 292

Mathematical Practices

1. How can you use a definition to construct an argument in Exercise 54 on page 280?
2. How is the form of the function you wrote in Exercise 52 on page 288 related to the forms of other types of functions you have learned about in this course?
3. What patterns did you use to solve Exercise 27 on page 296?

Analyzing Your Errors

Misreading Directions

- **What Happens**: You incorrectly read or do not understand directions.
- **How to Avoid This Error**: Read the instructions for exercises at least twice and make sure you understand what they mean. Make this a habit and use it when taking tests.
Graph the function. Describe the domain and range. (Section 6.1)

1. \( y = 5^x \)  
2. \( y = -2 \left( \frac{1}{6} \right)^x \)  
3. \( y = 6(2)^x - 4 - 1 \)

Determine whether the table represents an exponential growth function, an exponential decay function, or neither. Explain. (Section 6.2)

4. 
<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>7</td>
<td>21</td>
<td>63</td>
<td>189</td>
</tr>
</tbody>
</table>

5. 
<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>14,641</td>
<td>1331</td>
<td>121</td>
<td>11</td>
</tr>
</tbody>
</table>

Determine whether the function represents exponential growth or exponential decay. Identify the percent rate of change. (Section 6.2)

6. \( y = 3(1.88)^t \)  
7. \( f(t) = \frac{1}{3}(1.26)^t \)  
8. \( f(t) = 80 \left( \frac{3}{5} \right)^t \)

Tell whether the table of values represents a linear or an exponential function. Then write the function. (Section 6.3)

9. 
<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>48</td>
<td>12</td>
<td>3</td>
<td>( \frac{3}{4} )</td>
<td>( \frac{3}{16} )</td>
</tr>
</tbody>
</table>

10. 
<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>8</td>
<td>2</td>
<td>-4</td>
<td>-10</td>
<td>-16</td>
</tr>
</tbody>
</table>

11. The function \( f(t) = 5(4)^t \) represents the number of frogs in a pond after \( t \) years. (Section 6.1 and Section 6.2)
   a. Does the function represent exponential growth or exponential decay? Explain.
   b. Graph the function. Describe the domain and range.
   c. What is the yearly percent change?
   d. How many frogs are in the pond after 4 years?

12. You deposit $500 in a savings account that earns 3% annual interest compounded monthly. (Section 6.2)
   a. Write a function that represents the balance after \( t \) years.
   b. What is the balance of the account after 36 months? 10 years?

13. The table shows the amount \( y \) (in grams) of an element remaining in a jar after \( x \) centuries. (Section 6.1 and Section 6.3)

<table>
<thead>
<tr>
<th>Centuries, ( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount remaining, ( y )</td>
<td>50</td>
<td>25</td>
<td>12.5</td>
<td>6.25</td>
<td>3.125</td>
</tr>
</tbody>
</table>

   a. Plot the points. Let the number \( x \) of centuries represent the independent variable.
   b. Tell whether the data can be modeled by a linear or an exponential function. Then write the function.
   c. How much of the element remains after 600 years?
6.4 Solving Exponential Equations

**Essential Question** How can you solve an exponential equation graphically?

**Exploration 1** Solving an Exponential Equation Graphically

Work with a partner. Use a graphing calculator to solve the exponential equation $2.5^x - 3 = 6.25$ graphically. Describe your process and explain how you determined the solution.

**Exploration 2** The Number of Solutions of an Exponential Equation

Work with a partner.

a. Use a graphing calculator to graph the equation $y = 2^x$.

![Graph of $y = 2^x$]

b. In the same viewing window, graph a linear equation (if possible) that does not intersect the graph of $y = 2^x$.

c. In the same viewing window, graph a linear equation (if possible) that intersects the graph of $y = 2^x$ in more than one point.

d. Is it possible for an exponential equation to have no solution? more than one solution? Explain your reasoning.

**Exploration 3** Solving Exponential Equations Graphically

Work with a partner. Use a graphing calculator to solve each equation.

a. $2^x = \frac{1}{2}$  

b. $2^x + 1 = 0$  

c. $2^x = 1$

d. $3^x = 9$

e. $3^x - 1 = 0$

f. $4^{2x} = \frac{1}{16}$

g. $2^{3x} = \frac{1}{8}$  

h. $3^x + 2 = \frac{1}{9}$

i. $2^x - 2 = \frac{3}{2}x - 2$

**Communicate Your Answer**

4. How can you solve an exponential equation graphically?

5. A population of 30 mice is expected to double each year. The number $p$ of mice in the population each year is given by $p = 30(2^x)$. In how many years will there be 960 mice in the population?
6.4 Lesson

Core Vocabulary

exponential equation, p. 300

What You WillLearn

- Solve exponential equations with the same base.
- Solve exponential equations with unlike bases.
- Solve exponential equations by graphing.

Solving Exponential Equations with the Same Base

Exponential equations are equations in which variable expressions occur as exponents.

Core Concept

Property of Equality for Exponential Equations

<table>
<thead>
<tr>
<th>Words</th>
<th>Two powers with the same positive base $b$, where $b \neq 1$, are equal if and only if their exponents are equal.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers</td>
<td>If $2^x = 2^y$, then $x = y$. If $x = 5$, then $2^x = 2^5$.</td>
</tr>
<tr>
<td>Algebra</td>
<td>If $b &gt; 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.</td>
</tr>
</tbody>
</table>

EXAMPLE 1  Solving Exponential Equations with the Same Base

Solve each equation.

a. $3^x + 1 = 3^5$

   SOLUTION
   
   Write the equation.
   
   $x + 1 = 5$
   
   Equate the exponents.
   
   $-1 - 1$
   
   Subtract 1 from each side.
   
   $x = 4$
   
   Simplify.

b. $6^2 = 6^{2x - 3}$

   Write the equation.
   
   $1 = 2x - 3$
   
   Equate the exponents.
   
   $+3 +3$
   
   Add 3 to each side.
   
   $4 = 2x$
   
   Simplify.
   
   $2 = 2x$
   
   Divide each side by 2.
   
   $x = 2$
   
   Simplify.

c. $10^{3x} = 10^{2x + 3}$

   Write the equation.
   
   $3x = 2x + 3$
   
   Equate the exponents.
   
   $-2x -2x$
   
   Subtract $2x$ from each side.
   
   $x = 3$
   
   Simplify.

Monitoring Progress

Help in English and Spanish at BigIdeasMath.com

Solve the equation. Check your solution.

1. $2^{2x} = 2^6$
2. $5^{2x} = 5^{x + 1}$
3. $7^{3x} + 5 = 7^x + 1$
Solving Exponential Equations with Unlike Bases

To solve some exponential equations, you must first rewrite each side of the equation using the same base.

**EXAMPLE 2**  

**Solving Exponential Equations with Unlike Bases**

Solve (a) $5^x = 125$, (b) $4^x = 2^x - 3$, and (c) $9^x + 2 = 27^x$.

**SOLUTION**

a. $5^x = 125$
   
   Write the equation.

   $5^x = 5^3$
   
   Rewrite 125 as $5^3$.

   $x = 3$
   
   Equate the exponents.

b. $4^x = 2^x - 3$
   
   Write the equation.

   $(2^2)^x = 2^x - 3$
   
   Power of a Power Property

   $2^x = 2^x - 3$
   
   Equate the exponents.

   $2x = x - 3$
   
   Solve for $x$.

   $x = -3$

**Check**

<table>
<thead>
<tr>
<th>$4^{-3}$</th>
<th>$2^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{64}$</td>
<td>$\frac{1}{64}$</td>
</tr>
</tbody>
</table>

✓

EXAMPLE 3

**Solving Exponential Equations When $0 < b < 1$**

Solve (a) $(\frac{1}{2})^x = 4$ and (b) $4^x + 1 = \frac{1}{64}$.

**SOLUTION**

a. $(\frac{1}{2})^x = 4$
   
   Write the equation.

   $(2^{-1})^x = 2^2$
   
   Power of a Power Property

   $2^{-x} = 2^2$
   
   Equate the exponents.

   $-x = 2$
   
   Solve for $x$.

   $x = -2$

**Check**

<table>
<thead>
<tr>
<th>$4^{-2}$</th>
<th>$\frac{1}{16}$</th>
</tr>
</thead>
</table>

✓

b. $4^x + 1 = \frac{1}{64}$
   
   Write the equation.

   $4^x + 1 = \frac{1}{4^3}$
   
   Definition of negative exponent

   $4^x + 1 = 4^{-3}$
   
   Equate the exponents.

   $x + 1 = -3$
   
   Solve for $x$.

   $x = -4$

**Check**

| $\frac{1}{64}$ | $\frac{1}{64}$ |

✓

**Monitoring Progress**

Solve the equation. Check your solution.

4. $4^x = 256$
5. $9^x = 3^{x - 6}$
6. $4^{3x} = 8^{x + 1}$
7. $(\frac{1}{2})^{x - 1} = 27$
Solving Exponential Equations by Graphing

Sometimes, it is difficult or impossible to rewrite each side of an exponential equation using the same base. You can solve these types of equations by graphing each side and finding the point(s) of intersection. Exponential equations can have no solution, one solution, or more than one solution depending on the number of points of intersection.

**EXAMPLE 4** Solving Exponential Equations by Graphing

Use a graphing calculator to solve (a) \(2.4^{x - 1} = 5.76\) and (b) \(3^x + 2 = x + 1\).

**SOLUTION**

**a. Step 1** Write a system of equations using each side of the equation.

\[
\begin{align*}
y &= 2.4^{x - 1} & \text{Equation 1} \\
y &= 5.76 & \text{Equation 2}
\end{align*}
\]

Step 2 Enter the equations into a calculator. Then graph the equations in a viewing window that shows where the graphs could intersect.

Step 3 Use the *intersect* feature to find the point of intersection. The graphs intersect at \((3, 5.76)\).

So, the solution is \(x = 3\).

**b. Step 1** Write a system of equations using each side of the equation.

\[
\begin{align*}
y &= 3^x + 2 & \text{Equation 1} \\
y &= x + 1 & \text{Equation 2}
\end{align*}
\]

Step 2 Enter the equations into a calculator. Then graph the equations in a viewing window that shows where the graphs could intersect.

The graphs do not intersect. So, the equation has no solution.

**Monitoring Progress**

Use a graphing calculator to solve the equation.

8. \(3.1^{x + 2} = 9.61\)
9. \(4^x - 3 = 3x - 8\)
10. \((\frac{1}{4})^x = -2x - 3\)
6.4 Exercises

**Vocabulary and Core Concept Check**

1. **WRITING** Describe how to solve an exponential equation with unlike bases.

2. **WHICH ONE DOESN’T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

   - $2^x = 4x + 6$
   - $5^{3x + 8} = 5^{2x}$
   - $3^4 = x + 4^2$
   - $2^x - 7 = 2^7$

**Monitoring Progress and Modeling with Mathematics**

In Exercises 3–12, solve the equation. Check your solution. *(See Examples 1 and 2.)*

3. $4^{3x} = 4^{10}$
4. $7^{x - 4} = 7^8$
5. $3^{2x} = 3^{2x + 8}$
6. $2^{4x} = 2^x + 9$
7. $2^x = 64$
8. $3^x = 243$
9. $7^{x - 5} = 49^x$
10. $216^x = 6^{x + 10}$
11. $64^{2x + 4} = 16^{3x}$
12. $27^x = 9^{x - 2}$

In Exercises 13–18, solve the equation. Check your solution. *(See Example 3.)*

13. $(\frac{1}{3})^x = 125$
14. $(\frac{1}{4})^x = 256$
15. $\frac{1}{128} = 2^{5x + 3}$
16. $3^{4x - 9} = \frac{1}{243}$
17. $36^{3x + 3} = (\frac{1}{216})^{x + 1}$
18. $(\frac{1}{27})^{4 - x} = 9^{2x - 1}$

**ERROR ANALYSIS** In Exercises 19 and 20, describe and correct the error in solving the exponential equation.

19. $5^{3x + 2} = 25x - 8$
   - $3x + 2 = x - 8$
   - $x = -8$

20. $(\frac{1}{2})^{5x} = 32x + 8$
   - $(2^3)^{5x} = (2^5)^{x + 8}$
   - $2^{15x} = 2^{5x + 40}$
   - $15x = 5x + 40$
   - $x = 4$

In Exercises 21–24, match the equation with the graph that can be used to solve it. Then solve the equation.

21. $2^x = 4$
22. $4^{2x - 5} = 4$
23. $2^x + 2 = 4$
24. $2^{-x - 2} = 4$

C. 25. $0.25x + 2 = 16$

26. $1.9^x - 4 = 3.61$
27. $(\frac{1}{2})^{7x + 1} = -9$
28. $(\frac{1}{3})^{x + 3} = 9$
29. $2^{x + 3} = 3x + 8$
30. $4x - 3 = 5^{-x - 1}$
31. $\frac{4}{3}x - 1 = (\frac{1}{3})^{2x - 1}$
32. $2^{-x + 1} = \frac{19 - 15x}{4}$
33. $5^x = -4^{-x + 4}$
34. $7^{x - 2} = 2^{x - 2}$
35. $3^{x - 3} = 2^{-x + 3}$
36. $5^{-2x + 3} = -6^x + 5$

Section 6.4 Solving Exponential Equations
In Exercises 37–40, solve the equation by using the Property of Equality for Exponential Equations.

37. \(30 \cdot 5^{x+3} = 150\)  
38. \(12 \cdot 2^{x-7} = 24\)

39. \(4(3^{-2x} - 4) = 36\)  
40. \(2(4^{2x} + 1) = 128\)

41. **MODELING WITH MATHEMATICS** You scan a photo into a computer at four times its original size. You continue to increase its size repeatedly by 100% using the computer. The new size of the photo \(y\) in comparison to its original size after \(x\) enlargements on the computer is represented by \(y = 2^{x+2}\). How many times must the photo be enlarged on the computer so the new photo is 32 times the original size?

42. **MODELING WITH MATHEMATICS** A bacterial culture quadruples in size every hour. You begin observing the number of bacteria 3 hours after the culture is prepared. The amount \(y\) of bacteria \(x\) hours after the culture is prepared is represented by \(y = 192(4^{x-3})\). When will there be 196,608 bacteria?

In Exercises 43–46, solve the equation.

43. \(3^{4x} + 6 = 27^{x} + 2\)  
44. \(3^{4x} + 3 = 81^{x}\)

45. \(4^{x} + 3 = 2^{2(x+1)}\)  
46. \(5^{8(x-1)} = 625^{2x-2}\)

47. **NUMBER SENSE** Explain how you can use mental math to solve the equation \(8^{x-4} = 1\).

48. **PROBLEM SOLVING** There are a total of 128 teams at the start of a citywide 3-on-3 basketball tournament. Half the teams are eliminated after each round. Write and solve an exponential equation to determine after which round there are 16 teams left.

49. **PROBLEM SOLVING** You deposit $500 in an account that earns 6% annual interest compounded yearly. Write and solve an exponential equation to determine when the balance of the account will be $561.80.

50. **HOW DO YOU SEE IT?** The graph shows the annual attendance at two events. Each event began in 2004.

   ![Graph showing event attendance](image)

   a. Estimate when the events will have about the same attendance.
   b. Explain how you can verify your answer in part (a).

51. **REASONING** Explain why the Property of Equality for Exponential Equations does not work when \(b = 1\). Give an example to justify your answer.

52. **THOUGHT PROVOKING** Is it possible for an exponential equation to have two different solutions? If not, explain your reasoning. If so, give an example.

In Exercises 53–56, use a graphing calculator to solve the equation.

53. \(2^{x} = \sqrt{2}\)  
54. \(\sqrt{7} = 7^{-x}\)

55. \(8^{x-1/2} = \sqrt{8}\)  
56. \(\sqrt{5} = 5^{x+1}\)

57. **USING STRUCTURE** Use the results of Exercises 53–56 to find the value of \(x\) for which \(a^{x} = \sqrt{a}\).

58. **MAKING AN ARGUMENT** Consider the equation \((\frac{1}{a})^{x} = b\), where \(a > 1\) and \(b > 1\). Your friend says the value of \(x\) will always be negative. Is your friend correct? Explain.

### Maintaining Mathematical Proficiency

Determine whether the sequence is arithmetic. If so, find the common difference. *(Section 4.6)*

59. \(-20, -26, -32, -38, \ldots\)  
60. \(9, 18, 36, 72, \ldots\)

61. \(-5, -8, -12, -17, \ldots\)  
62. \(10, 20, 30, 40, \ldots\)

304 Chapter 6 Exponential Functions and Sequences
6.5 Geometric Sequences

**Essential Question** How can you use a geometric sequence to describe a pattern?

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**.

**EXPLORATION 1** Describing Calculator Patterns

Work with a partner. Enter the keystrokes on a calculator and record the results in the table. Describe the pattern.

a. Step 1
   - \[ \text{Step 1} \times 2 = \]
   - \[ \text{Step 2} \times 2 = \]
   - \[ \text{Step 3} \times 2 = \]
   - \[ \text{Step 4} \times 2 = \]
   - \[ \text{Step 5} \times 2 = \]

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculator display</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Step 1
   - \[ \text{Step 1} \times 2 = \]
   - \[ \text{Step 2} \times 2 = \]
   - \[ \text{Step 3} \times 2 = \]
   - \[ \text{Step 4} \times 2 = \]
   - \[ \text{Step 5} \times 2 = \]

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculator display</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Use a calculator to make your own sequence. Start with any number and multiply by 3 each time. Record your results in the table.

d. Part (a) involves a geometric sequence with a common ratio of 2. What is the common ratio in part (b)? part (c)?

**EXPLORATION 2** Folding a Sheet of Paper

Work with a partner. A sheet of paper is about 0.1 millimeter thick.

a. How thick will it be when you fold it in half once? twice? three times?

b. What is the greatest number of times you can fold a piece of paper in half? How thick is the result?

c. Do you agree with the statement below? Explain your reasoning.

“If it were possible to fold the paper in half 15 times, it would be taller than you.”

**Communicate Your Answer**

3. How can you use a geometric sequence to describe a pattern?

4. Give an example of a geometric sequence from real life other than paper folding.
### What You Will Learn

- Identify geometric sequences.
- Extend and graph geometric sequences.
- Write geometric sequences as functions.

### Identifying Geometric Sequences

#### Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.

\[
1, 5, 25, 125, \ldots \quad \text{Terms of a geometric sequence}
\]

\[
\times 5 \quad \times 5 \quad \times 5 \quad \text{common ratio}
\]

#### EXAMPLE 1 Identifying Geometric Sequences

Decide whether each sequence is **arithmetic**, **geometric**, or **neither**. Explain your reasoning.

**a.** 120, 60, 30, 15, \ldots

**b.** 2, 6, 11, 17, \ldots

**SOLUTION**

**a.** Find the ratio between each pair of consecutive terms.

\[
\begin{align*}
120 & \quad 60 & \quad 30 & \quad 15 \\
60 & \quad 30 & \quad 15 & \quad 30
\end{align*}
\]

\[
\frac{60}{120} = \frac{1}{2} \quad \frac{30}{60} = \frac{1}{2} \quad \frac{15}{30} = \frac{1}{2}
\]

The ratios are the same. The common ratio is \(\frac{1}{2}\).

- So, the sequence is geometric.

**b.** Find the ratio between each pair of consecutive terms.

\[
\begin{align*}
2 & \quad 6 & \quad 11 & \quad 17 \\
6 & \quad 11 & \quad 17
\end{align*}
\]

\[
\frac{6}{2} = 3 \quad \frac{11}{6} = \frac{17}{17}
\]

There is no common ratio, so the sequence is **not** geometric.

Find the difference between each pair of consecutive terms.

\[
\begin{align*}
2 & \quad 6 & \quad 11 & \quad 17 \\
6 & \quad 11 & \quad 17
\end{align*}
\]

\[
\begin{align*}
6 - 2 &= 4 \\
11 - 6 &= 5 \\
17 - 11 &= 6
\end{align*}
\]

There is no common difference, so the sequence is **not** arithmetic.

- So, the sequence is **neither** geometric nor arithmetic.

### Monitoring Progress

Decide whether the sequence is **arithmetic**, **geometric**, or **neither**. Explain your reasoning.

1. 5, 1, −3, −7, \ldots
2. 1024, 128, 16, 2, \ldots
3. 2, 6, 10, 16, \ldots
Extending and Graphing Geometric Sequences

**EXAMPLE 2** Extending Geometric Sequences

Write the next three terms of each geometric sequence.

a. 3, 6, 12, 24, . . .

b. 64, −16, 4, −1, . . .

**SOLUTION**

Use tables to organize the terms and extend each sequence.

a. The next three terms are 48, 96, and 192.

b. The next three terms are \(\frac{1}{4}\), \(-\frac{1}{16}\), and \(-\frac{1}{64}\).

**LOOKING FOR STRUCTURE**

When the terms of a geometric sequence alternate between positive and negative terms, or vice versa, the common ratio is negative.

**STUDY TIP**

The points of any geometric sequence with a positive common ratio lie on an exponential curve.

**EXAMPLE 3** Graphing a Geometric Sequence

Graph the geometric sequence 32, 16, 8, 4, 2, . . .. What do you notice?

**SOLUTION**

Make a table. Then plot the ordered pairs \((n, a_n)\).

<table>
<thead>
<tr>
<th>Position, (n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term, (a_n)</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

The points appear to lie on an exponential curve.

**Monitoring Progress**

Write the next three terms of the geometric sequence. Then graph the sequence.

4. 1, 3, 9, 27, . . .

5. 2500, 500, 100, 20, . . .

6. 80, −40, 20, −10, . . .

7. −2, 4, −8, 16, . . .
Writing Geometric Sequences as Functions

Because consecutive terms of a geometric sequence have a common ratio, you can use the first term $a_1$ and the common ratio $r$ to write an exponential function that describes a geometric sequence. Let $a_1 = 1$ and $r = 5$.

<table>
<thead>
<tr>
<th>Position, $n$</th>
<th>Term, $a_n$</th>
<th>Written using $a_1$ and $r$</th>
<th>Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>first term, $a_1$</td>
<td>$a_1$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>second term, $a_2$</td>
<td>$a_1r$</td>
<td>$1 \cdot 5 = 5$</td>
</tr>
<tr>
<td>3</td>
<td>third term, $a_3$</td>
<td>$a_1r^2$</td>
<td>$1 \cdot 5^2 = 25$</td>
</tr>
<tr>
<td>4</td>
<td>fourth term, $a_4$</td>
<td>$a_1r^3$</td>
<td>$1 \cdot 5^3 = 125$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$n$th term, $a_n$</td>
<td>$a_1r^{n-1}$</td>
<td>$1 \cdot 5^{n-1}$</td>
</tr>
</tbody>
</table>

STUDY TIP
Notice that the equation $a_n = a_1r^{n-1}$ is of the form $y = ab^n$.

**Equation for a Geometric Sequence**
Let $a_n$ be the $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$. The $n$th term is given by $a_n = a_1r^{n-1}$.

**EXAMPLE 4** Finding the $n$th Term of a Geometric Sequence

Write an equation for the $n$th term of the geometric sequence 2, 12, 72, 432, . . .. Then find $a_{10}$.

**SOLUTION**

The first term is 2, and the common ratio is 6.

$$a_n = a_1r^{n-1}$$
Equation for a geometric sequence

$$a_n = 2(6)^{n-1}$$
Substitute 2 for $a_1$ and 6 for $r$.

Use the equation to find the 10th term.

$$a_n = 2(6)^{n-1}$$
Write the equation.

$$a_{10} = 2(6)^{10-1}$$
Substitute 10 for $n$.

$$= 20,155,392$$
Simplify.

The 10th term of the geometric sequence is 20,155,392.

**Monitoring Progress**

Write an equation for the $n$th term of the geometric sequence. Then find $a_7$.

8. 1, −5, 25, −125, . . .
9. 13, 26, 52, 104, . . .
10. 432, 72, 12, 2, . . .
11. 4, 10, 25, 62.5, . . .
You can rewrite the equation for a geometric sequence with first term $a_1$ and common ratio $r$ in function notation by replacing $a_n$ with $f(n)$.

$$f(n) = a_1 r^{n-1}$$

The domain of the function is the set of positive integers.

**EXAMPLE 5** Modeling with Mathematics

Clicking the zoom-out button on a mapping website doubles the side length of the square map. After how many clicks on the zoom-out button is the side length of the map 640 miles?

**SOLUTION**

1. **Understand the Problem** You know that the side length of the square map doubles after each click on the zoom-out button. So, the side lengths of the map represent the terms of a geometric sequence. You need to find the number of clicks it takes for the side length of the map to be 640 miles.

2. **Make a Plan** Begin by writing a function $f$ for the $n$th term of the geometric sequence. Then find the value of $n$ for which $f(n) = 640$.

3. **Solve the Problem**

   The first term is 5, and the common ratio is 2.

   $$f(n) = a_1 r^{n-1} \quad \text{Function for a geometric sequence}$$

   $$f(n) = 5(2)^{n-1} \quad \text{Substitute 5 for } a_1 \text{ and 2 for } r.$$  

   The function $f(n) = 5(2)^{n-1}$ represents the geometric sequence. Use this function to find the value of $n$ for which $f(n) = 640$. So, use the equation $640 = 5(2)^{n-1}$ to write a system of equations.

   $$y = 5(2)^{n-1} \quad \text{Equation 1}$$
   $$y = 640 \quad \text{Equation 2}$$

   Then use a graphing calculator to graph the equations and find the point of intersection. The point of intersection is $(8, 640)$.

   So, after eight clicks, the side length of the map is 640 miles.

4. **Look Back** Find the value of $n$ for which $f(n) = 640$ algebraically.

   $$640 = 5(2)^{n-1} \quad \text{Write the equation.}$$
   $$128 = (2)^{n-1} \quad \text{Divide each side by 5.}$$
   $$2^7 = (2)^{n-1} \quad \text{Rewrite 128 as } 2^7.$$  
   $$7 = n - 1 \quad \text{Equate the exponents.}$$
   $$8 = n \quad \text{Add 1 to each side.}$$

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12. **WHAT IF?** After how many clicks on the zoom-out button is the side length of the map 2560 miles?
6.5 Exercises

Vocabulary and Core Concept Check

1. **Writing** Compare the two sequences.
   
   
   \[2, 4, 6, 8, 10, \ldots\] \[2, 4, 8, 16, 32, \ldots\]

2. **Critical Thinking** Why do the points of a geometric sequence lie on an exponential curve only when the common ratio is positive?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, find the common ratio of the geometric sequence.

3. 4, 12, 36, 108, \ldots
4. 36, 6, 1, \frac{1}{2}, \ldots
5. \frac{3}{5}, -3, 24, -192, \ldots
6. 0.1, 1, 10, 100, \ldots
7. 128, 96, 72, 54, \ldots
8. -162, 54, -18, 6, \ldots

In Exercises 9–14, determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning. (See Example 1.)

9. -8, 0, 8, 16, \ldots
10. -1, 4, -7, 10, \ldots
11. 9, 14, 20, 27, \ldots
12. \frac{3}{2}, \frac{3}{4}, 3, 21, \ldots
13. 192, 24, 3, \frac{3}{2}, \ldots
14. -25, -18, -11, -4, \ldots

In Exercises 15–18, determine whether the graph represents an arithmetic sequence, a geometric sequence, or neither. Explain your reasoning.

15. \(a_n\)
16. \(a_n\)

In Exercises 19–24, write the next three terms of the geometric sequence. Then graph the sequence. (See Examples 2 and 3.)

19. 5, 20, 80, 320, \ldots
20. -3, 12, -48, 192, \ldots
21. 81, -27, 9, -3, \ldots
22. -375, -75, -15, -3, \ldots
23. 32, 8, 2, \frac{1}{2}, \ldots
24. \frac{16}{9}, \frac{8}{3}, 4, 6, \ldots

In Exercises 25–32, write an equation for the \(n\)th term of the geometric sequence. Then find \(a_6\). (See Example 4.)

25. 2, 8, 32, 128, \ldots
26. 0.6, -3, 15, -75, \ldots
27. -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{2}, -1, \ldots
28. 0.1, 0.9, 8.1, 72.9, \ldots
29. 

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>7640</td>
<td>764</td>
<td>76.4</td>
<td>7.64</td>
</tr>
</tbody>
</table>

30. 

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>-192</td>
<td>48</td>
<td>-12</td>
<td>3</td>
</tr>
</tbody>
</table>

31. 

32. 

33. **Problem Solving** A badminton tournament begins with 128 teams. After the first round, 64 teams remain. After the second round, 32 teams remain. How many teams remain after the third, fourth, and fifth rounds?
34. **PROBLEM SOLVING** The graphing calculator screen displays an area of 96 square units. After you zoom out once, the area is 384 square units. After you zoom out a second time, the area is 1536 square units. What is the screen area after you zoom out four times?

![Graphing Calculator Screen](image)

35. **ERROR ANALYSIS** Describe and correct the error in writing the next three terms of the geometric sequence.

\[ -8, 4, -2, 1, \ldots \]

The next three terms are \(-2, 4, \text{and} -8\).

36. **ERROR ANALYSIS** Describe and correct the error in writing an equation for the \(n\)th term of the geometric sequence.

\[ -2, -12, -72, -432, \ldots \]

The first term is \(-2\), and the common ratio is \(-6\).

\[ a_n = a_1 r^{n-1} \]

\[ a_n = -2(-6)^{n-1} \]

37. **MODELING WITH MATHEMATICS** The distance (in millimeters) traveled by a swinging pendulum decreases after each swing, as shown in the table.

(See Example 5.)

<table>
<thead>
<tr>
<th>Swing</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (in millimeters)</td>
<td>625</td>
<td>500</td>
<td>400</td>
</tr>
</tbody>
</table>

a. Write a function that represents the distance the pendulum swings on its \(n\)th swing.

b. After how many swings is the distance 256 millimeters?

38. **MODELING WITH MATHEMATICS** You start a chain email and send it to six friends. The next day, each of your friends forwards the email to six people. The process continues for a few days.

a. Write a function that represents the number of people who have received the email after \(n\) days.

b. After how many days will 1296 people have received the email?

39. **MODELING WITH MATHEMATICS** In Exercises 39 and 40, (a) write a function that represents the sequence of figures and (b) describe the \(10\)th figure in the sequence.

39. 

40. 

41. **REASONING** Write a sequence that represents the number of teams that have been eliminated after \(n\) rounds of the badminton tournament in Exercise 33. Determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning.

42. **REASONING** Write a sequence that represents the perimeter of the graphing calculator screen in Exercise 34 after you zoom out \(n\) times. Determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning.

43. **WRITING** Compare the graphs of arithmetic sequences to the graphs of geometric sequences.

44. **MAKING AN ARGUMENT** You are given two consecutive terms of a sequence.

\[ \ldots, -8, 0, \ldots \]

Your friend says that the sequence is not geometric. A classmate says that is impossible to know given only two terms. Who is correct? Explain.
45. CRITICAL THINKING Is the sequence shown an arithmetic sequence? a geometric sequence? Explain your reasoning.

3, 3, 3, 3, . . .

46. HOW DO YOU SEE IT? Without performing any calculations, match each equation with its graph. Explain your reasoning.

a. \( a_n = 20 \left( \frac{4}{3} \right)^{n-1} \)

b. \( a_n = 20 \left( \frac{3}{4} \right)^{n-1} \)

<table>
<thead>
<tr>
<th>A.</th>
<th>B.</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Graph A]</td>
<td>![Graph B]</td>
</tr>
</tbody>
</table>

47. REASONING What is the 9th term of the geometric sequence where \( a_3 = 81 \) and \( r = 3 \)?

48. OPEN-ENDED Write a sequence that has a pattern but is not arithmetic or geometric. Describe the pattern.

49. ATTENDING TO PRECISION Are the terms of a geometric sequence independent or dependent? Explain your reasoning.

50. DRAWING CONCLUSIONS A college student makes a deal with her parents to live at home instead of living on campus. She will pay her parents \$0.01 for the first day of the month, \$0.02 for the second day, \$0.04 for the third day, and so on.

a. Write an equation that represents the \( n \)th term of the geometric sequence.

b. What will she pay on the 25th day?

c. Did the student make a good choice or should she have chosen to live on campus? Explain.

51. REPEATED REASONING A soup kitchen makes 16 gallons of soup. Each day, a quarter of the soup is served and the rest is saved for the next day.

a. Write the first five terms of the sequence of the number of fluid ounces of soup left each day.

b. Write an equation that represents the \( n \)th term of the sequence.

c. When is all the soup gone? Explain.

52. THOUGHT PROVOKING Find the sum of the terms of the geometric sequence.

\( \frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n - 1}, \ldots \)

Explain your reasoning. Write a different infinite geometric sequence that has the same sum.

53. OPEN-ENDED Write a geometric sequence in which \( a_2 < a_1 < a_3 \).

54. NUMBER SENSE Write an equation that represents the \( n \)th term of each geometric sequence shown.

\[
\begin{array}{cccc}
\hline
n & 1 & 2 & 3 & 4 \\
\hline
a_n & 2 & 6 & 18 & 54 \\
\hline
b_n & 1 & 5 & 25 & 125 \\
\hline
\end{array}
\]

a. Do the terms \( a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots \) form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

b. Do the terms \( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \) form a geometric sequence? If so, how does the common ratio relate to the common ratios of the sequences above?

55. Use residuals to determine whether the model is a good fit for the data in the table. Explain. (Section 4.5)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-10</td>
<td>-2</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

56. \( y = -5x + 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>-4</td>
<td>-3</td>
</tr>
</tbody>
</table>

Chapter 6 Exponential Functions and Sequences
6.6 Recursively Defined Sequences

**Essential Question** How can you define a sequence recursively?

A recursive rule gives the beginning term(s) of a sequence and a recursive equation that tells how \( a_n \) is related to one or more preceding terms.

**EXPLORATION 1** Describing a Pattern

**Work with a partner.** Consider a hypothetical population of rabbits. Start with one breeding pair. After each month, each breeding pair produces another breeding pair. The total number of rabbits each month follows the exponential pattern 2, 4, 8, 16, 32, \ldots. Now suppose that in the first month after each pair is born, the pair is too young to reproduce. Each pair produces another pair after it is 2 months old. Find the total number of pairs in months 6, 7, and 8.

<table>
<thead>
<tr>
<th>Month</th>
<th>Number of pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

**RECOGNIZING PATTERNS**

To be proficient in math, you need to look closely to discern a pattern or structure.

**EXPLORATION 2** Using a Recursive Equation

**Work with a partner.** Consider the following recursive equation.

\[
a_n = a_{n-1} + a_{n-2}
\]

Each term in the sequence is the sum of the two preceding terms.

Copy and complete the table. Compare the results with the sequence of the number of pairs in Exploration 1.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_4 )</td>
<td>( a_5 )</td>
<td>( a_6 )</td>
<td>( a_7 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Communicate Your Answer**

3. How can you define a sequence recursively?

4. Use the Internet or some other reference to determine the mathematician who first described the sequences in Explorations 1 and 2.
What You Will Learn

- Write terms of recursively defined sequences.
- Write recursive rules for sequences.
- Translate between recursive rules and explicit rules.
- Write recursive rules for special sequences.

Writing Terms of Recursively Defined Sequences

So far in this book, you have defined arithmetic and geometric sequences explicitly. An explicit rule gives an as a function of the term’s position number n in the sequence. For example, an explicit rule for the arithmetic sequence 3, 5, 7, 9, … is $a_n = 3 + 2(n - 1)$, or $a_n = 2n + 1$.

Now, you will define arithmetic and geometric sequences recursively. A recursive rule gives the beginning term(s) of a sequence and a recursive equation that tells how $a_n$ is related to one or more preceding terms.

Core Concept

Recursive Equation for an Arithmetic Sequence

$a_n = a_{n-1} + d$, where $d$ is the common difference

Recursive Equation for a Geometric Sequence

$a_n = r \cdot a_{n-1}$, where $r$ is the common ratio

Example 1

Writing Terms of Recursively Defined Sequences

Write the first six terms of each sequence. Then graph each sequence.

a. $a_1 = 2$, $a_n = a_{n-1} + 3$

b. $a_1 = 1$, $a_n = 3a_{n-1}$

Solution

You are given the first term. Use the recursive equation to find the next five terms.

a. $a_1 = 2$

- $a_2 = a_1 + 3 = 2 + 3 = 5$
- $a_3 = a_2 + 3 = 5 + 3 = 8$
- $a_4 = a_3 + 3 = 8 + 3 = 11$
- $a_5 = a_4 + 3 = 11 + 3 = 14$
- $a_6 = a_5 + 3 = 14 + 3 = 17$

b. $a_1 = 1$

- $a_2 = 3a_1 = 3(1) = 3$
- $a_3 = 3a_2 = 3(3) = 9$
- $a_4 = 3a_3 = 3(9) = 27$
- $a_5 = 3a_4 = 3(27) = 81$
- $a_6 = 3a_5 = 3(81) = 243$

STUDY TIP

A sequence is a discrete function. So, the points on the graph are not connected.
Monitoring Progress

Write the first six terms of the sequence. Then graph the sequence.

1. \(a_1 = 0, a_n = a_{n-1} - 8\)
2. \(a_1 = -7.5, a_n = a_{n-1} + 2.5\)
3. \(a_1 = -36, a_n = \frac{1}{2}a_{n-1}\)
4. \(a_1 = 0.7, a_n = 10a_{n-1}\)

Writing Recursive Rules

EXAMPLE 2 Writing Recursive Rules

Write a recursive rule for each sequence.

a. \(-30, -18, -6, 6, 18, \ldots\)

SOLUTION

Use a table to organize the terms and find the pattern.

\[
\begin{array}{c|c|c|c|c|c}
\text{Position, } n & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{Term, } a_n & -30 & -18 & -6 & 6 & 18 \\
\hline
\end{array}
\]

The sequence is arithmetic, with first term \(a_1 = -30\) and common difference \(d = 12\).

\[
a_n = a_{n-1} + d \quad \text{Recursive equation for an arithmetic sequence}
\]

So, a recursive rule for the sequence is \(a_1 = -30, a_n = a_{n-1} + 12\).

b. \(500, 100, 20, 4, 0.8, \ldots\)

\[
\begin{array}{c|c|c|c|c|c}
\text{Position, } n & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{Term, } a_n & 500 & 100 & 20 & 4 & 0.8 \\
\hline
\end{array}
\]

The sequence is geometric, with first term \(a_1 = 500\) and common ratio \(r = \frac{1}{5}\).

\[
a_n = r \cdot a_{n-1} \quad \text{Recursive equation for a geometric sequence}
\]

So, a recursive rule for the sequence is \(a_1 = 500, a_n = \frac{1}{5}a_{n-1}\).

Monitoring Progress

Write a recursive rule for the sequence.

5. \(8, 3, -2, -7, -12, \ldots\)
6. \(1.3, 2.6, 3.9, 5.2, 6.5, \ldots\)
7. \(4, 20, 100, 500, 2500, \ldots\)
8. \(128, -32, 8, -2, 0.5, \ldots\)
9. Write a recursive rule for the height of the sunflower over time.
Translating between Recursive and Explicit Rules

**EXAMPLE 3** Translating from Recursive Rules to Explicit Rules

Write an explicit rule for each recursive rule.

a. \( a_1 = 25, a_n = a_{n-1} - 10 \)

b. \( a_1 = 19.6, a_n = -0.5a_{n-1} \)

**SOLUTION**

a. The recursive rule represents an arithmetic sequence, with first term \( a_1 = 25 \) and common difference \( d = -10 \).

\[
a_n = a_1 + (n - 1)d
\]

Explicit rule for an arithmetic sequence

\[
a_n = 25 + (n - 1)(-10)
\]

Substitute 25 for \( a_1 \) and \(-10\) for \( d \).

\[
a_n = -10n + 35
\]

Simplify.

An explicit rule for the sequence is \( a_n = -10n + 35 \).

b. The recursive rule represents a geometric sequence, with first term \( a_1 = 19.6 \) and common ratio \( r = -0.5 \).

\[
a_n = a_1r^{n-1}
\]

Explicit rule for a geometric sequence

\[
a_n = 19.6(-0.5)^{n-1}
\]

Substitute 19.6 for \( a_1 \) and \(-0.5\) for \( r \).

An explicit rule for the sequence is \( a_n = 19.6(-0.5)^{n-1} \).

**EXAMPLE 4** Translating from Explicit Rules to Recursive Rules

Write a recursive rule for each explicit rule.

a. \( a_n = -2n + 3 \)

b. \( a_n = -3(2)^{n-1} \)

**SOLUTION**

a. The explicit rule represents an arithmetic sequence, with first term \( a_1 = -2(1) + 3 = 1 \) and common difference \( d = -2 \).

\[
a_n = a_{n-1} + d
\]

Recursive equation for an arithmetic sequence

\[
a_n = a_{n-1} + (-2)
\]

Substitute \(-2\) for \( d \).

So, a recursive rule for the sequence is \( a_1 = 1, a_n = a_{n-1} - 2 \).

b. The explicit rule represents a geometric sequence, with first term \( a_1 = -3 \) and common ratio \( r = 2 \).

\[
a_n = r \cdot a_{n-1}
\]

Recursive equation for a geometric sequence

\[
a_n = 2a_{n-1}
\]

Substitute \(2\) for \( r \).

So, a recursive rule for the sequence is \( a_1 = -3, a_n = 2a_{n-1} \).

**Monitoring Progress**

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Write an explicit rule for the recursive rule.

10. \( a_1 = -45, a_n = a_{n-1} + 20 \)

11. \( a_1 = 13, a_n = -3a_{n-1} \)

Write a recursive rule for the explicit rule.

12. \( a_n = -n + 1 \)

13. \( a_n = -2.5(4)^{n-1} \)
Writing Recursive Rules for Special Sequences

You can write recursive rules for sequences that are neither arithmetic nor geometric. One way is to look for patterns in the sums of consecutive terms.

**EXAMPLE 5** Writing Recursive Rules for Other Sequences

Use the sequence shown.

1, 1, 2, 3, 5, 8, . . .

**a.** Write a recursive rule for the sequence.

**b.** Write the next three terms of the sequence.

**SOLUTION**

**a.** Find the difference and ratio between each pair of consecutive terms.

<table>
<thead>
<tr>
<th>Term</th>
<th>Difference</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

There is no common difference, so the sequence is not arithmetic.

**b.** Find the sum of each pair of consecutive terms.

\[
\begin{align*}
a_1 + a_2 &= 1 + 1 = 2 \quad & \text{2 is the third term.} \\
a_2 + a_3 &= 1 + 2 = 3 \quad & \text{3 is the fourth term.} \\
a_3 + a_4 &= 2 + 3 = 5 \quad & \text{5 is the fifth term.} \\
a_4 + a_5 &= 3 + 5 = 8 \quad & \text{8 is the sixth term.}
\end{align*}
\]

Beginning with the third term, each term is the sum of the two previous terms. A recursive equation for the sequence is \(a_n = a_{n-2} + a_{n-1}\).

Then, a recursive rule for the sequence is

\[
a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-2} + a_{n-1}.
\]

**b.** Use the recursive equation \(a_n = a_{n-2} + a_{n-1}\) to find the next three terms.

\[
\begin{align*}
a_7 &= a_5 + a_6 \quad & = 5 + 8 = 13 \\
a_8 &= a_6 + a_7 \quad & = 8 + 13 = 21 \\
a_9 &= a_7 + a_8 \quad & = 13 + 21 = 34
\end{align*}
\]

The next three terms are 13, 21, and 34.

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Write a recursive rule for the sequence. Then write the next three terms of the sequence.

**14.** 5, 6, 11, 17, 28, . . .

**15.** -3, -4, -7, -11, -18, . . .

**16.** 1, 1, 0, -1, -1, 0, 1, 1, . . .

**17.** 4, 3, 1, 2, -1, 3, -4, . . .
Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A recursive rule gives the beginning term(s) of a sequence and a(n) ______________ that tells how \( a_n \) is related to one or more preceding terms.

2. **WHICH ONE DOESN'T BELONG?** Which rule does not belong with the other three? Explain your reasoning.

   \[ a_1 = -1, \ a_n = 5a_{n-1} \quad a_n = 6n - 2 \quad a_1 = -3, \ a_n = a_{n-1} + 1 \quad a_1 = 9, \ a_n = 4a_{n-1} \]

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine whether the recursive rule represents an arithmetic sequence or a geometric sequence.

3. \( a_1 = 2, \ a_n = 7a_{n-1} \)
4. \( a_1 = 18, \ a_n = a_{n-1} + 1 \)
5. \( a_1 = 5, \ a_n = a_{n-1} - 4 \)
6. \( a_1 = 3, \ a_n = -6a_{n-1} \)

In Exercises 7–12, write the first six terms of the sequence. Then graph the sequence. (See Example 1.)

7. \( a_1 = 0, \ a_n = a_{n-1} + 2 \)
8. \( a_1 = 10, \ a_n = a_{n-1} - 5 \)
9. \( a_1 = 2, \ a_n = 3a_{n-1} \)
10. \( a_1 = 8, \ a_n = 1.5a_{n-1} \)
11. \( a_1 = 80, \ a_n = -\frac{1}{2}a_{n-1} \)
12. \( a_1 = -7, \ a_n = -4a_{n-1} \)

In Exercises 13–20, write a recursive rule for the sequence. (See Example 2.)

13. \[
\begin{array}{c|cccc}
   n & \_ & \_ & \_ & \_ \\
   a_n & 7 & 16 & 25 & 34 \\
\end{array}
\]
14. \[
\begin{array}{c|cccc}
   n & \_ & \_ & \_ & \_ \\
   a_n & 8 & 24 & 72 & 216 \\
\end{array}
\]
15. 243, 81, 27, 9, 3, . . .
16. 3, 11, 19, 27, 35, . . .

17. 0, −3, −6, −9, −12, . . .
18. 5, −20, 80, −320, 1280, . . .
19. \[
\begin{array}{c|c}
   n & a_n \\
   \_ & \_ \\
\end{array}
\]
20. \[
\begin{array}{c|c}
   n & a_n \\
   \_ & \_ \\
\end{array}
\]

21. **MODELING WITH MATHEMATICS** Write a recursive rule for the number of bacterial cells over time.

22. **MODELING WITH MATHEMATICS** Write a recursive rule for the length of the deer antler over time.
In Exercises 23–28, write an explicit rule for the recursive rule. (See Example 3.)

23. \( a_1 = -3, a_n = a_{n-1} + 3 \)
24. \( a_1 = 8, a_n = a_{n-1} - 12 \)
25. \( a_1 = 16, a_n = 0.5a_{n-1} \)
26. \( a_1 = -2, a_n = 9a_{n-1} \)
27. \( a_1 = 4, a_n = a_{n-1} + 17 \)
28. \( a_1 = 5, a_n = -5a_{n-1} \)

In Exercises 29–34, write a recursive rule for the explicit rule. (See Example 4.)

29. \( a_n = 7(3)^{n-1} \)
30. \( a_n = -4n + 2 \)
31. \( a_n = 1.5n + 3 \)
32. \( a_n = 6n - 20 \)
33. \( a_n = (-5)^n - 1 \)
34. \( a_n = -81 \left( \frac{2}{3} \right)^{n-1} \)

In Exercises 35–38, graph the first four terms of the sequence with the given description. Write a recursive rule and an explicit rule for the sequence.

35. The first term of a sequence is 5. Each term of the sequence is 15 more than the preceding term.
36. The first term of a sequence is 16. Each term of the sequence is half the preceding term.
37. The first term of a sequence is -1. Each term of the sequence is -3 times the preceding term.
38. The first term of a sequence is 19. Each term of the sequence is 13 less than the preceding term.

In Exercises 39–44, write a recursive rule for the sequence. Then write the next two terms of the sequence. (See Example 5.)

39. 1, 3, 4, 7, 11, . . .
40. 10, 9, 1, 8, -7, 15, . . .
41. 2, 4, 2, -2, -4, -2, . . .
42. 6, 1, 7, 8, 15, 23, . . .
43. \[ a_n \]
44. \[ a_n \]

45. ERROR ANALYSIS Describe and correct the error in writing an explicit rule for the recursive rule \( a_1 = 6, a_n = a_{n-1} - 12 \).

46. ERROR ANALYSIS Describe and correct the error in writing a recursive rule for the sequence 2, 4, 6, 10, 16, . . ..

47. \( f(1) = 3, f(n) = f(n - 1) + 7 \)
48. \( f(1) = -1, f(n) = 6f(n - 1) \)
49. \( f(1) = 8, f(n) = -f(n - 1) \)
50. \( f(1) = 4, f(2) = 5, f(n) = f(n - 2) + f(n - 1) \)
51. \( f(1) = 10, f(2) = 15, f(n) = f(n - 1) - f(n - 2) \)

52. MODELING WITH MATHEMATICS The X-ray shows the lengths (in centimeters) of bones in a human hand.

a. Write a recursive rule for the lengths of the bones.

b. Measure the lengths of different sections of your hand. Can the lengths be represented by a recursively defined sequence? Explain.

Section 6.6  Recursively Defined Sequences  319
53. USING TOOLS You can use a spreadsheet to generate the terms of a sequence.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>=</td>
<td>A1+2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

a. To generate the terms of the sequence \( a_1 = 3, a_n = a_{n-1} + 2 \), enter the value of \( a_1, 3 \), into cell A1. Then enter “=A1+2” into cell A2, as shown. Use the fill down feature to generate the first 10 terms of the sequence.

b. Use a spreadsheet to generate the first 10 terms of the sequence \( a_1 = 3, a_n = 4a_{n-1} \). (Hint: Enter “=4*A1” into cell A2.)

c. Use a spreadsheet to generate the first 10 terms of the sequence \( a_1 = 4, a_2 = 7, a_n = a_{n-1} - a_{n-2} \). (Hint: Enter “=A2-A1” into cell A3.)

54. HOW DO YOU SEE IT? Consider Squares 1–6 in the diagram.

a. Write a sequence in which each term \( a_n \) is the side length of square \( n \).

b. What is the name of this sequence? What is the next term of this sequence?

c. Use the term in part (b) to add another square to the diagram and extend the spiral.

55. REASONING Write the first 5 terms of the sequence \( a_1 = 5, a_n = 3a_{n-1} + 4 \). Determine whether the sequence is arithmetic, geometric, or neither. Explain your reasoning.

56. THOUGHT PROVOKING Describe the pattern for the numbers in Pascal’s Triangle, shown below. Write a recursive rule that gives the \( m \)th number in the \( n \)th row.

\[
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

57. REASONING The explicit rule \( a_n = a_1 + (n - 1)d \) defines an arithmetic sequence.

a. Explain why \( a_{n-1} = a_1 + [(n - 1) - 1]d \).

b. Justify each step in showing that a recursive equation for the sequence is \( a_n = a_{n-1} + d \).

\[
\begin{align*}
a_n &= a_1 + (n - 1)d \\
&= a_1 + [(n - 1) + 0]d \\
&= a_1 + [(n - 1) - 1 + 1]d \\
&= a_1 + [(n - 1) - 1] + 1\]d \\
&= a_1 + [(n - 1) - 1]d + d \\
&= a_{n-1} + d
\end{align*}
\]

58. MAKING AN ARGUMENT Your friend claims that the sequence 

\[-5, 5, -5, 5, -5, \ldots \]

cannot be represented by a recursive rule. Is your friend correct? Explain.

59. PROBLEM SOLVING Write a recursive rule for the sequence.

\[3, 7, 15, 31, 63, \ldots \]

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. (Skills Review Handbook)

60. \( 5x + 12x \)  
61. \( 9 - 6y - 14 \)  
62. \( 2d - 7 - 8d \)  
63. \( 3 - 3m + 11m \)

Write a linear function \( f \) with the given values. (Section 4.2)

64. \( f(2) = 6, f(-1) = -3 \)  
65. \( f(-2) = 0, f(6) = -4 \)  
66. \( f(-3) = 5, f(-1) = 5 \)  
67. \( f(3) = -1, f(-4) = -15 \)
6.4–6.6 What Did You Learn?

Core Vocabulary

- exponential equation, p. 300
- geometric sequence, p. 306
- common ratio, p. 306
- explicit rule, p. 314
- recursive rule, p. 314

Core Concepts

Section 6.4

- Property of Equality for Exponential Equations, p. 300
- Solving Exponential Equations by Graphing, p. 302

Section 6.5

- Geometric Sequence, p. 306
- Equation for a Geometric Sequence, p. 308

Section 6.6

- Recursive Equation for an Arithmetic Sequence, p. 314
- Recursive Equation for a Geometric Sequence, p. 314

Mathematical Practices

1. How did you decide on an appropriate level of precision for your answer in Exercise 50 part (a) on page 304?
2. Explain how writing a function in Exercise 39 part (a) on page 311 created a shortcut for answering part (b).
3. How did you choose an appropriate tool in Exercise 52 part (b) on page 319?

Performance Task:

Mathematical Recursion

Most people think of the Fibonacci sequence when they think about mathematical recursion. How are recursive sequences used in language, art, music, nature, and games?

To explore the answer to this question and more, check out the Performance Task and Real-Life STEM video at BigIdeasMath.com.
Chapter Review

6.1 Exponential Functions (pp. 273–280)

Graph \( f(x) = 9(3)^x \).

**Step 1** Make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
</tr>
</tbody>
</table>

**Step 2** Plot the ordered pairs.

**Step 3** Draw a smooth curve through the points.

Graph the function. Describe the domain and range.

1. \( f(x) = -4 \left( \frac{1}{2} \right)^x \)
2. \( f(x) = 3x^2 + 2 \)
3. \( f(x) = 2^x - 4 - 3 \)

4. Write and graph an exponential function \( f \) represented by the table. Then compare the graph to the graph of \( g(x) = \left( \frac{1}{2} \right)^x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>0.25</td>
</tr>
</tbody>
</table>

6.2 Exponential Growth and Decay (pp. 281–288)

Determine whether \( y = 2(1.13)^t \) represents exponential growth or exponential decay. Identify the percent rate of change.

The function is of the form \( y = a(1 + r)^t \), where \( 1 + r > 1 \), so it represents exponential growth.

Use the growth factor \( 1 + r \) to find the rate of growth.

\[
1 + r = 1.13 \\
\therefore r = 0.13
\]

Write an equation. 

Solve for \( r \).

So, the function represents exponential growth and the rate of growth is 13%.

Determine whether the table represents an exponential growth function, an exponential decay function, or neither. Explain.

5. \[
\begin{align*}
\text{ } x & \quad 0 \quad 1 \quad 2 \quad 3 \\
\text{ } y & \quad 3 \quad 6 \quad 12 \quad 24 \\
\end{align*}
\]

6. \[
\begin{align*}
\text{ } x & \quad 1 \quad 2 \quad 3 \quad 4 \\
\text{ } y & \quad 162 \quad 108 \quad 72 \quad 48 \\
\end{align*}
\]

Determine whether the function represents exponential growth or exponential decay. Identify the percent rate of change.

7. \( y = 0.99^t \)
8. \( f(t) = 6(0.84)^t \)
9. \( f(t) = 4(1.05)^t \)

10. You deposit $750 in a savings account that earns 5% annual interest compounded quarterly. (a) Write a function that represents the balance after \( t \) years. (b) What is the balance of the account after 4 years?

11. The value of a TV is $1500. Its value decreases by 14% each year. (a) Write a function that represents the value \( y \) (in dollars) of the TV after \( t \) years. (b) What is the value of the TV after 3 years? Round your answer to the nearest dollar.
### 6.3 Comparing Linear and Exponential Functions (pp. 289–296)

Tell whether each table of values represents a linear or an exponential function. Then write the function.

**a.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-6</td>
<td>-1</td>
<td>4</td>
<td>9</td>
<td>14</td>
</tr>
</tbody>
</table>

The differences of consecutive $y$-values are constant. The slope is $\frac{5}{1} = 5$ and the $y$-intercept is $-1$. So, the table represents the linear function $y = 5x - 1$.

**b.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

Consecutive $y$-values have a common ratio of 2 and the $y$-intercept is 20. So, the table represents the exponential function $y = 20(2)^x$.

Tell whether the table represents a linear or an exponential function. Then write the function.

**12.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>384</td>
<td>96</td>
<td>24</td>
<td>6</td>
<td>1.5</td>
</tr>
</tbody>
</table>

**13.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>64</td>
<td>48</td>
<td>32</td>
<td>16</td>
<td>0</td>
</tr>
</tbody>
</table>

**14.** The balance $y$ (in dollars) of your savings account after $t$ years is represented by $y = 300(1.05)^t$. The beginning balance of your friend’s account is $340, and the balance increases by $10 each year. (a) Compare the account balances by calculating and interpreting the average rates of change from $t = 2$ to $t = 6$. (b) Predict which account will have a greater balance after 10 years. Explain.

### 6.4 Solving Exponential Equations (pp. 299–304)

Solve $\frac{1}{9} = 3^x + 6$.

Write the equation.

$\frac{1}{9} = 3^x + 6$

Rewrite $\frac{1}{9}$ as $3^{-2}$.

$3^{-2} = 3^x + 6$

Equate the exponents.

$-2 = x + 6$

Solve for $x$.

$x = -8$

**15.** $5^x = 5^{x-2}$

**16.** $3^x - 2 = 1$

**17.** $-4 = 6^{4x - 3}$

**18.** $(\frac{1}{3})^{2x + 3} = x + 5$

**19.** $(\frac{1}{16})^{3x} = 64^{2(x + 8)}$

**20.** $27^{2x + 2} = 81^{x + 4}$
6.5 Geometric Sequences (pp. 305–312)

Write the next three terms of the geometric sequence 2, 6, 18, 54, . . . .

Use a table to organize the terms and extend the sequence.

<table>
<thead>
<tr>
<th>Position</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>2</td>
<td>6</td>
<td>18</td>
<td>54</td>
<td>162</td>
<td>486</td>
<td>1458</td>
</tr>
</tbody>
</table>

Each term is 3 times the previous term. So, the common ratio is 3.

The next three terms are 162, 486, and 1458.

Decide whether the sequence is arithmetic, geometric, or neither. Explain your reasoning. If the sequence is geometric, write the next three terms and graph the sequence.

21. 3, 12, 48, 192, . . . 22. 9, −18, 27, −36, . . . 23. 375, −75, 15, −3, . . .

Write an equation for the $n$th term of the geometric sequence. Then find $a_9$.

24. 1, 4, 16, 64, . . .
25. 5, −10, 20, −40, . . .
26. 486, 162, 54, 18, . . .

6.6 Recursively Defined Sequences (pp. 313–320)

Write a recursive rule for the sequence 5, 12, 19, 26, 33, . . .

Use a table to organize the terms and find the pattern.

<table>
<thead>
<tr>
<th>Position, $n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term, $a_n$</td>
<td>5</td>
<td>12</td>
<td>19</td>
<td>26</td>
<td>33</td>
</tr>
</tbody>
</table>

The sequence is arithmetic, with first term $a_1 = 5$ and common difference $d = 7$.

\[ a_n = a_{n-1} + d \quad \text{Recursive equation for an arithmetic sequence} \]

\[ a_n = a_{n-1} + 7 \quad \text{Substitute 7 for } d. \]

So, a recursive rule for the sequence is $a_1 = 5, a_n = a_{n-1} + 7$.

Write the first six terms of the sequence. Then graph the sequence.

27. $a_1 = 4, a_n = a_{n-1} + 5$ 28. $a_1 = -4, a_n = -3a_{n-1}$ 29. $a_1 = 32, a_n = \frac{1}{4}a_{n-1}$

Write a recursive rule for the sequence.

30. 3, 8, 13, 18, 23, . . . 31. 3, 6, 12, 24, 48, . . . 32. 7, 6, 13, 19, 32, . . .

33. The first term of a sequence is 8. Each term of the sequence is 5 times the preceding term. Graph the first four terms of the sequence. Write a recursive rule and an explicit rule for the sequence.
Write and graph a function that represents the situation.

1. Your starting annual salary of $42,500 increases by 3% each year.
2. You deposit $500 in an account that earns 6.5% annual interest compounded yearly.

Tell whether the table of values represents a linear or an exponential function. Then write the function.

3. \[ \begin{array}{c|c|c|c|c} x & 0 & 1 & 2 & 3 \\ \hline y & -8 & -4 & 0 & 4 \\ \end{array} \]

4. \[ \begin{array}{c|c|c|c|c} x & -2 & -1 & 0 & 1 \\ \hline y & 0.2 & 0.6 & 1.8 & 5.4 \\ \end{array} \]

Write an explicit rule and a recursive rule for the sequence.

5. \[ \begin{array}{c|c|c|c} n & 1 & 2 & 3 \\ \hline a_n & -6 & 8 & 22 \\ \end{array} \]

6. \[ \begin{array}{c|c|c|c} n & 1 & 2 & 3 \\ \hline a_n & 400 & 100 & 25 \\ \end{array} \]

Solve the equation. Check your solution.

7. \[ 2^x = \frac{1}{128} \]
8. \[ 256^x + 2 = 16^{3x-1} \]

9. Graph \( f(x) = 2(6)^x \). Compare the graph to the graph of \( g(x) = 6^x \). Describe the domain and range of \( f \).

Determine whether the function represents exponential growth or exponential decay. Identify the percent rate of change.

10. \( y = 1.7(1.09)^x \)
11. \( r(t) = 4\left(\frac{1}{3}\right)^t \)

12. The first two terms of a sequence are \( a_1 = 3 \) and \( a_2 = -12 \). Let \( a_3 \) be the third term when the sequence is arithmetic and let \( b_3 \) be the third term when the sequence is geometric. Find \( a_3 - b_3 \).

13. At sea level, Earth’s atmosphere exerts a pressure of 1 atmosphere. Atmospheric pressure \( P \) (in atmospheres) decreases with altitude. It can be modeled by \( P = (0.99988)^a \), where \( a \) is the altitude (in meters).

   a. Identify the initial amount, decay factor, and decay rate.

   b. Use a graphing calculator to graph the function. Use the graph to estimate the atmospheric pressure at an altitude of 5000 feet.

14. You follow the training schedule from your coach.

   a. Write an explicit rule and a recursive rule for the geometric sequence.

   b. What is the first day that you run more than 2 kilometers? more than 3 kilometers?

   c. What is the total distance that you run in the first week? Round your answer to the nearest kilometer.
1. Fill in values to write a linear function \( g \) so that \( f(x) = g(x) \) has two solutions, \( x = 2 \) and \( x = 4 \).

\[
f(x) = \frac{1}{2}(2)^x \quad \quad g(x) = \underline{x} \quad \underline{-} \quad \underline{2}
\]

2. The graph of the exponential function \( f \) is shown. Find \( f(-7) \).

3. Student A claims he can form a linear system from the equations shown that has infinitely many solutions. Student B claims she can form a linear system from the equations shown that has one solution. Student C claims he can form a linear system from the equations shown that has no solution.

\[
\begin{align*}
3x + y &= 12 \\
3x + 2y &= 12 \\
6x + 2y &= 6 \\
3y + 9x &= 36 \\
2y - 6x &= 12 \\
9x - 3y &= -18
\end{align*}
\]

**a.** Select two equations to support Student A’s claim.

**b.** Select two equations to support Student B’s claim.

**c.** Select two equations to support Student C’s claim.

4. Fill in the inequality with <, ≤, >, or ≥ so that the system of linear inequalities has no solution.

**Inequality 1** \( y - 2x \leq 4 \)

**Inequality 2** \( 6x - 3y \underline{\quad} - 12 \)

5. The second term of a sequence is 7. Each term of the sequence is 10 more than the preceding term. Fill in values to write a recursive rule and an explicit rule for the sequence.

\[
a_1 = \underline{7}, \quad a_n = a_{n-1} + \underline{10} \\
a_n = \underline{10n} - \underline{3}
\]

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6. A data set consists of the heights \( y \) (in feet) of a hot-air balloon \( t \) minutes after it begins its descent. An equation of the line of best fit is \( y = 870 - 14.8t \). Which of the following is a correct interpretation of the line of best fit?

A. The initial height of the hot-air balloon is 870 feet. The slope has no meaning in this context.

B. The initial height of the hot-air balloon is 870 feet, and it descends 14.8 feet per minute.

C. The initial height of the hot-air balloon is 870 feet, and it ascends 14.8 feet per minute.

D. The hot-air balloon descends 14.8 feet per minute. The \( y \)-intercept has no meaning in this context.

7. Select all the functions whose \( x \)-value is an integer when \( f(x) = 10 \).

\[
\begin{align*}
\text{Exponential} & \\
\text{Linear} & \\
\text{Neither} & \\
\hline
f(x) = -2(8)^x & f(x) = 15 - x & f(x) = \frac{1}{2}(3)^x \\
f(x) = 6x^2 + 9 & f(x) = 4(1.6)^x & f(x) = x(18 - x) \\
f(x) = 3(\frac{1}{6})^x & f(x) = -3(4x + 1 - x) & f(x) = 2
\end{align*}
\]

8. Place each function into one of the three categories. For exponential functions, state whether the function represents exponential growth, exponential decay, or neither.

9. How does the graph shown compare to the graph of \( f(x) = 2^x \)?

\[
\begin{align*}
\text{Exponential} & \\
\text{Linear} & \\
\text{Neither} & \\
\hline
f(x) = -2(8)^x & f(x) = 15 - x & f(x) = \frac{1}{2}(3)^x \\
f(x) = 6x^2 + 9 & f(x) = 4(1.6)^x & f(x) = x(18 - x) \\
f(x) = 3(\frac{1}{6})^x & f(x) = -3(4x + 1 - x) & f(x) = 2
\end{align*}
\]